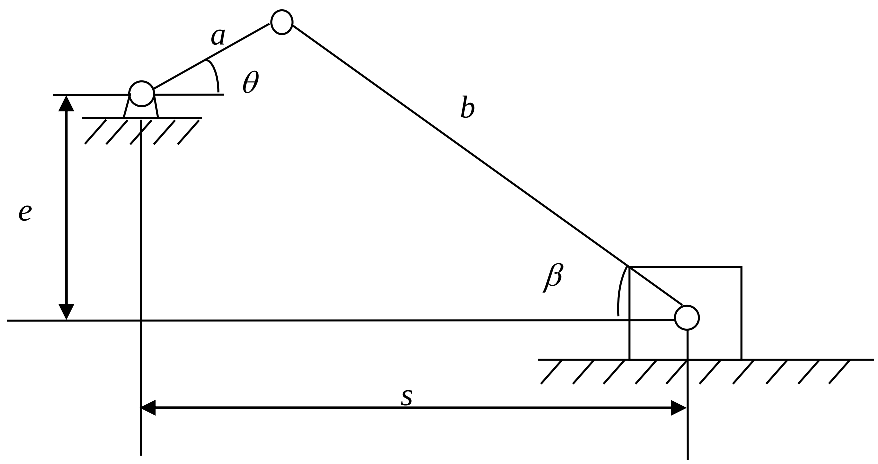
## Example 1

The position of a slider-crank mechanism is required to be $s_a = 35$ mm when $\theta = 10^\circ$. If the actual position $s$ is outside the tolerance range $s_a \pm 0.1$ mm, it is considered that a failure occurs. The design variables are independent and follow normal distributions of $a \sim N(11.3, 0.067^2)\,\text{mm}$, $b \sim N(25.3, 0.02^2)\,\text{mm}$, and $e \sim N(6.5, 0.01^2)\,\text{mm}$. Use the First Order Second Moment (FOSM) method to calculate the probability of failure.




**Solution:**

The vector of random variables is $\mathbf{X} = (a, b, e)$. The actual position of the slider is given by

$$s = a\cos\theta + \sqrt{b^2 - (e + a\sin\theta)^2}$$

Since the tolerance range is $s_a \pm 0.1$, failure occurs when $s > s_a + 0.1$ or $s < s_a - 0.1$. Let $Y_1 = g_1(\mathbf{X}) = s_a + 0.1 - s$, and $Y_2 = g_2(\mathbf{X}) = s - (s_a - 0.1)$. Then, the probability of failure of the mechanism is given by

$$p_f = \Pr(Y_1 < 0) + \Pr(Y_2 < 0)$$

Since the design variables are independent and follow normal distributions, $Y_1$ and $Y_2$ also follow normal distributions with $Y_1 \sim N(\mu_{Y_1}, \sigma_{Y_1}^2)$ and $Y_2 \sim N(\mu_{Y_2}, \sigma_{Y_2}^2)$, respectively.

According to the FOSM method, the mean and standard deviation of $Y_1$ are calculated by

$$\mu_{Y_1} = g_1(\boldsymbol{\mu}_{\mathbf{X}}) = (s_a + 0.1) - \left(\mu_{\mathrm{a}} \cos\theta + \sqrt{(\mu_b)^2 - (\mu_e + \mu_{\mathrm{a}} \sin\theta)^2}\right) = 0.1288\,\mathrm{mm}$$

$$\frac{\partial g_1}{\partial a} = -\left(\cos\theta + \frac{1}{2}\left(\frac{1}{\sqrt{b^2 - (e + a\sin\theta)^2}}\right)\left(-2(e + a\sin\theta)\right)(\sin\theta)\right)\Bigg|_{(b=\mu_b,\, e=\mu_e,\, a=\mu_a)} = -0.9232$$

$$\frac{\partial g_1}{\partial b} = -\left(\frac{1}{2}\left(\frac{1}{\sqrt{b^2 - (e + a\sin\theta)^2}}\right)(2b)\right)\Bigg|_{(b=\mu_b,\, e=\mu_e,\, a=\mu_a)} = -1.0611$$

$$\frac{\partial g_1}{\partial e} = -\left(\frac{1}{2}\left(\frac{1}{\sqrt{b^2 - (e + a\sin\theta)^2}}\right)\left(-2(e + a\sin\theta)\right)\right)\Bigg|_{(b=\mu_b,\, e=\mu_e,\, a=\mu_a)} = 0.3549$$

$$\sigma_{Y_1} = \sqrt{\left(\frac{\partial g_1}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial g_1}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial g_1}{\partial e}\right)^2 \sigma_e^2}$$
$$= \sqrt{\left(-0.9232\right)^2 (0.067)^2 + \left(-1.0611\right)^2 \left(0.02\right)^2 + \left(0.3549\right)^2 \left(0.01\right)^2} = 0.0224\,\mathrm{mm}$$

Thus, we have

$$\Pr(Y_1 < 0) = \Phi\left(-\frac{\mu_{Y_1}}{\sigma_{Y_1}}\right) = \Phi(-5.7568) = 4.2849 \times 10^{-9}$$

Similarly, the mean and standard deviation of $Y_2$ are calculated by

$$\mu_{Y_2} = g_2(\boldsymbol{\mu}_{\mathbf{X}}) = \left(\mu_{\mathrm{a}} \cos\theta + \sqrt{(\mu_b)^2 - (\mu_e + \mu_{\mathrm{a}} \sin\theta)^2}\right) - (s_a - 0.1) = 0.0712\,\mathrm{mm}$$

$$\frac{\partial g_2}{\partial a} = \left(\cos\theta + \frac{1}{2}\left(\frac{1}{\sqrt{b^2 - (e + a\sin\theta)^2}}\right)\left(-2(e + a\sin\theta)\right)(\sin\theta)\right)\Bigg|_{(b=\mu_b,\, e=\mu_e,\, a=\mu_a)} = 0.9232$$

$$\frac{\partial g_2}{\partial b} = \left(\frac{1}{2}\left(\frac{1}{\sqrt{b^2 - (e + a\sin\theta)^2}}\right)(2b)\right)\Bigg|_{(b=\mu_b,\, e=\mu_e,\, a=\mu_a)} = 1.0611$$

$$\frac{\partial g_2}{\partial e} = \left(\frac{1}{2}\left(\frac{1}{\sqrt{b^2 - (e + a\sin\theta)^2}}\right)\left(-2(e + a\sin\theta)\right)\right)\Bigg|_{(b=\mu_b,\, e=\mu_e,\, a=\mu_a)} = -0.3549$$

$$\sigma_{Y_2} = \sqrt{\left(\frac{\partial g_2}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial g_2}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial g_2}{\partial e}\right)^2 \sigma_e^2}$$

$$= \sqrt{(0.9232)^2 (0.067)^2 + (1.0611)^2 (0.02)^2 + (-0.3549)^2 (0.01)^2} = 0.0224\,\text{mm}$$

We have

$$\Pr(Y_2 < 0) = \Phi\left(-\frac{\mu_{Y_2}}{\sigma_{Y_2}}\right) = \Phi(-3.1797) = 7.371\times10^{-4}$$

The probability of failure of this mechanism is then given by

$$p_f = \Pr(Y_1 < 0) + \Pr(Y_2 < 0) = 4.2849\times10^{-9} + 7.371\times10^{-4} = 7.371\times10^{-4}$$ **Ans.**