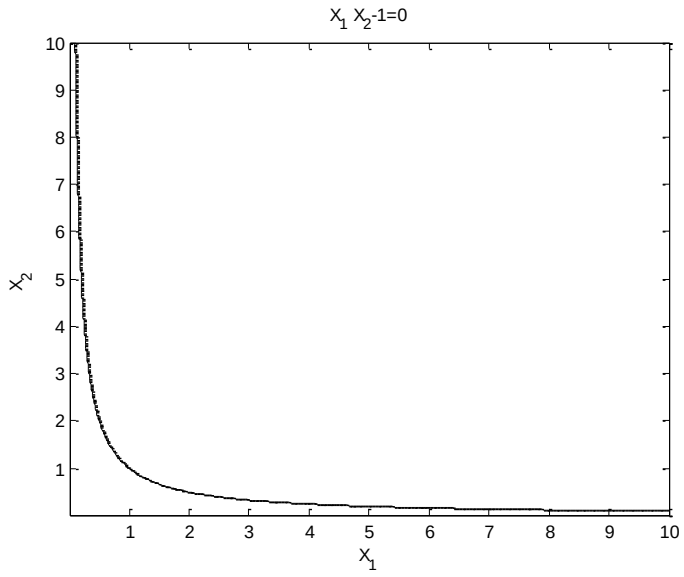
3. A limit-state function is given by $Y = g(X_1, X_2) = X_1X_2 - 1$, where $X_1 \sim N(5,2^2)$ and $X_2 \sim N(5,1^2)$. If $X_1$ and $X_2$ are independent, use FORM to find the probability of failure.




**Solution**

(1) Transformation from **X** to **U**

$$Fx_i(X_i) = \Phi(U_i), i = 1,2$$

$$\Phi\left(\frac{X_i - \mu_i}{\sigma_i}\right) = \Phi(U_i)$$

$$\frac{X_i - \mu_i}{\sigma_i} = U_i$$

$$X_i = \mu_i + \sigma_i U_i$$

$$X_1 = 5 + 2u_1, X_2 = 5 + u_2$$

$$\mathrm{g}(\mathbf{U}) = (5 + 2u_1)(5 + u_2) - 1$$

(2) MPP Search

1st iteration: $k = 0$

$$\boldsymbol{\nabla} g(\mathbf{U}) = \left(\frac{\partial \mathrm{g}}{\partial U_1}, \frac{\partial \mathrm{g}}{\partial U_2}\right) = (10 + 2U_2, 5 + 2U_1)$$

$$\mathbf{u}^{(0)} = (0,0)$$

$g(\mathbf{u}^{(0)}) = 24$

$\beta^{(0)} = \|\mathbf{u}^{(0)}\| = 0$

$\nabla g(\mathbf{u}^{(0)}) = (10,5)$

$\|\nabla g(\mathbf{u}^{(0)})\| = \sqrt{10^2 + 5^2} = 11.18$

$\boldsymbol{\alpha}^{(0)} = \dfrac{\nabla g(\mathbf{u}^{(0)})}{\|\nabla g(\mathbf{u}^{(0)})\|} = \dfrac{(10,5)}{11.18} = (0.8944,0.4472)$

2nd iteration: $k = 1$

$g(\mathbf{u}^{(0)}) = 24$

$\nabla g(\mathbf{u}^{(0)}) =$ （10,5）

$\beta^{(1)} = \beta^{(0)} + \dfrac{g(\mathbf{u}^{(0)})}{\|\nabla g(\mathbf{u}^{(0)})\|} = 2.1466$

$\boldsymbol{u}^{(1)} = -\boldsymbol{\alpha}^{(0)}\beta^{(1)} = -(0.8944,0.1421)(2.1466) = -(1.92,-0.96)$

3 rd iteration: $k = 2$

$\nabla g(\boldsymbol{u}^{(1)}) = (8.08,1.16)$

$\|\nabla g(\mathbf{u}^{(1)})\| = 8.1628$

$\boldsymbol{\alpha}^{(1)} = \dfrac{\nabla g(\mathbf{u}^{(1)})}{\|\nabla g(\mathbf{u}^{(1)})\|} = (0.9899,0.1421)$

$\beta^{2} = \beta^{(1)} + \dfrac{g(\mathbf{u}^{(1)})}{\|\nabla g(\mathbf{u}^{(1)})\|} = 2.5982$

$\mathbf{u}^{(2)} = -\boldsymbol{\alpha}^{(1)}\beta^{(2)} = -(0.9899,0.1421)(2.5982) = (-2.5720,-0.3692)$

We repeat the process until convergence. The detailed search history is shown in the following table.

**Table 1 Convergence History**

| $k$ | $u_1$ | $u_1$ | $g(\mathbf{U})$ | $\frac{\partial \mathrm{g}}{\partial U_1}$ | $\frac{\partial \mathrm{g}}{\partial U_2}$ | $\|\nabla g(\mathbf{U})\|$ | $\alpha_1$ | $\alpha_2$ | $\beta$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 0.0 | 24.0 | 10.0 | 5.0 | 11.1803 | 0.8944 | 0.4472 | 0.0 |
| 1 | -1.920 | -0.960 | 3.6866 | 8.080 | 1.160 | 8.1628 | 0.9899 | 0.1421 | 2.1466 |
| 2 | -2.5720 | -0.3692 | -1.6656 | 9.2615 | -0.1437 | 9.2627 | 0.9999 | -0.0155 | 2.5982 |
| 3 | -2.4181 | 0.0375 | -0.1751 | 10.0751 | 0.1637 | 10.0764 | 0.9999 | 0.0163 | 2.4184 |
| 4 | -2.4007 | -0.0390 | -0.0150 | 9.9220 | 0.1986 | 9.9239 | 0.9998 | 0.020 | 2.4010 |
| 5 | -2.3991 | -0.0480 | -0.0002 | 9.9040 | 0.2019 | 9.9060 | 0.9998 | 0.0204 | 2.3995 |
| 6 | -2.3990 | -0.0489 | 0.0 | 9.9022 | 0.2020 | 9.9042 | 0.9998 | 0.0204 | 2.3995 |

(3) Estimate $p_f$

$$p_f = \Phi(-\beta) = \beta(-2.3995) = 0.0082$$