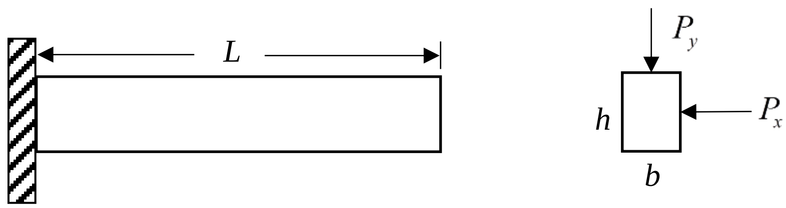
## Reliability-Based Design (RBD)

A cantilever beam is subjected to two random forces $P_x \sim N(500, 100^2)$ lb and $P_y \sim N(500, 100^2)$ N at the tip as shown. The yield stress is $S_y \sim N(40 \times 10^3, (2 \times 10^3)^2)$ psi, the Young's Modulus is $E \sim N(29 \times 10^6, (3 \times 10^6)^2)$ psi, and the allowable deflection is $d_0 = 2.5$ in. The length of the beam is a random variable with $L \sim N(\mu_L, 0.1^2)$ in. Given the minimum probabilities of $10^{-3}$ for failure modes of excessive normal stress and excessive deflection, find the optimal solution so that the volume of the beam is minimized. The ranges of the beam size are given by $1$ in $\leq b, h \leq 20$ in and $50$ in $\leq \mu_L \leq 100$ in.




**Solution**

The design variable are $\mathbf{d} = (\mathrm{b}, \mathrm{h})$ and $\mathbf{X} = (L)$; the objective function is $f(b, h, \mu_L) = bh\mu_L$. The random parameters are $\mathbf{P} = (P_x, P_y, S_y, E)$. There are two failure modes. For the failure mode of excessive stress, the limit-state function is given by

$$G_1 = \frac{6L}{bh}\left(\frac{P_x}{b} + \frac{P_y}{h}\right) - S_y \leq 0$$

For the failure mode of excessive deflection, the limit-state function is given by

$$G_2 = d_0 - \frac{4L^3}{E}\sqrt{\left(\frac{P_x}{b^3 h}\right)^2 + \left(\frac{P_y}{bh^3}\right)^2} \leq 0$$

All the information we know are listed below.

$P_x \sim N(500, 100^2)$ lb

$P_y \sim N(500, 100^2)$ lb

$S_y \sim N(40 \times 10^3, (2 \times 10^3)^2)$ psi

$E \sim N(29 \times 10^6, 3 \times 10^6)$ psi

$L \sim N(\mu_L, 0.1^2)$ in

$d_0 = 2.5$ in

$1$ in $\leq b \leq 20$ in, $1$ in $\leq h \leq 20$ in, $50$ in $\leq \mu_L \leq 100$ in

$[R_1] = [R_2] = 0.999$, or $[p_{f1}] = [p_{f2}] = 0.001$

In order to find the optimal solution, we minimize $f(d, h, \mu_L)$ with two reliability constraints. The optimization model is given by

$$\begin{cases} \underbrace{\min}_{DV=(b,h,\mu_L)} \quad f = bh\mu_L \\ \Pr\left\{G_1 = \frac{6L}{bh}\left(\frac{P_x}{b} + \frac{P_y}{h}\right) - S_y \le 0\right\} \le [p_{f1}] \\ \Pr\left\{G_2 = d_0 - \frac{4L^3}{E}\sqrt{\left(\frac{P_x}{b^3h}\right)^2 + \left(\frac{P_y}{bh^3}\right)^2} \le 0\right\} \le [p_{f1}] \\ 1 \le b \le 20 \\ 1 \le h \le 20 \\ 50 \le \mu_L \le 100 \end{cases}$$

The First Order Second Moment method (FOSM) is used for reliability analysis, and Matlab is used to solve the optimization model. The optimal solution is as follows:

Design variables: $(b, h, \mu_L) = (1.95, 3.09, 50)$ in

Objective function: $f(b, h, \mu_L) = 301.67 \text{ in}^2$

Probability of failure for failure mode 1: $p_{f1} = 0.001$

Probability of failure for failure mode 2: $p_{f2} = 0$

The compare RBD with deterministic optimization (only mean values are used), we give the results from both methods in the following table.

| Deterministic Optimization | RBD |
|---|---|
| Optimal point = $(b, h, L)$ = (1.55, 3.11, 50) in | Optimal point = $(b, h, L)$ = (1.95, 3.09, 50) in |
| Objective = Volume = 241.37 in$^2$ | Objective = Volume = 301.67 in$^2$ |
| $p_{f1} = 0.5$ | $p_{f1} = 0.001$ |
| $p_{f2} = 0$ | $p_{f2} = 0$ |

The Matlab codes are given below.

1) Main function

```
% Main program of RDB for a cantilever beam
% Xiaoping Du, Missouri S&T, 04/10/2017

clear all; warning off;
%----------------------- Reliability-Based Design---------------
dv0 = [2.0,3.0,40.0];                   % dvs=[b,h,uL]
dvl = [1.0,1.0,50.0];                   % lower bounds of dvs
dvu = [20.0,20.0,100.0];                % upper bounds of dvs
```

```
required_pf(1:2) = 0.001;    % Required probabilities of failure
options = optimset('Display','iter','MaxFunEvals',1000); %
Display convergence history
dv =
fmincon('rbd_obj_fun',dv0,[],[],[],[],dvl,dvu,'rbd_con_fun',opti
ons,required_pf);
%obj_fun: function for objective
%con_fun: function for constraints

%Posterior analysis
obj = rbd_obj_fun(dv);
[g,ceq] = rbd_con_fun(dv,required_pf); %g=pf-required_pf
pf = g+required_pf;
%Display the results
disp('-------------------RBD Results-------------------');
disp(['The optimal point = ',num2str(dv)]);
disp (['The objective function = ',num2str(obj)]);
for i = 1:2
    disp(['Probability of failure mode ',num2str(i),' = ',...
            num2str(pf(i))]);
end
```

2) Objective function

```
%Objective function
function f = rbd_obj_fun(dv,required_pf)
%dvs=[b,h,uL]
f = dv(1)*dv(2)*dv(3); %Volume
```

3) Constraint function

```
% Constraint functions
% FOSM is used to calculate the probability of failure
function [g,ceq] = rbd_con_fun(dv,required_pf)
ceq = [];             %No equality constraints
%d=(b,h), X=(L), P=(Px,Py,S,E)
%dv=[b,h,uL];
b = dv(1); h = dv(2); L = dv(3);
Px = 500; Py = 1000; S = 4.0e4; E = 29.0e6; %Mean
sL = 0.1; sPx = 100; sPy = 100; sS = 2.0e3; sE = 3.0e6; %Std
%Calculate mean of g
ug1 = S-6*L/b/h*(Px/b+Py/h);
ug2 = 2.5-4*L^3/E*((Px/b^3/h)^2+(Py/b/h^3)^2)^0.5;
%Calculate std of g
dg1dL = 6/b/h*(Px/b+Py/h);
dg1dPx = 6*L/b/h/b;
```

```
dg1dPy = 6*L/b/h/h;
dg1dE = 0;
dg1dS = 1;
dg2dL = -12*L^2/E*(Px^2/b^6/h^2+Py^2/b^2/h^6)^(1/2);
dg2dPx = -4*L^3/E/(Px^2/b^6/h^2+Py^2/b^2/h^6)^(1/2)*Px/b^6/h^2;
dg2dPy = -4*L^3/E/(Px^2/b^6/h^2+Py^2/b^2/h^6)^(1/2)*Py/b^2/h^6;
dg2dE = 4*L^3/E^2*(Px^2/b^6/h^2+Py^2/b^2/h^6)^(1/2);
dg2dS = 0;
stdg1 =
((dg1dL*sL)^2+(dg1dPx*sPx)^2+(dg1dPy*sPy)^2+(dg1dE*sE)^2+(dg1dS*
sS)^2)^0.5;
stdg2 =
((dg2dL*sL)^2+(dg2dPx*sPx)^2+(dg2dPy*sPy)^2+(dg2dE*sE)^2+(dg2dS*
sS)^2)^0.5;
%Calculate reliability
pf(1) = normcdf(-ug1/stdg1);
pf(2) = normcdf(-ug2/stdg2);
%Constraints
g(1) = pf(1)-required_pf(1); %Matlab uses g<=0
g(2) = pf(2)-required_pf(2);
```