Charter Two

Basic Probability Theory

2.1 Introduction

We have discussed the causes and effects of uncertainty in engineering systems in Chapter 1. In the remaining chapters, we will deal with uncertainty at three levels – uncertainty modeling, uncertainty analysis, and design under uncertainty. Uncertainty modeling will be introduced from Chapters 2 through 6. The task of uncertainty modeling is to quantify uncertainty with mathematical structures. The primary mathematical tool for uncertainty modeling is probability theory. The outcome of uncertainty modeling will then be used for uncertainty analysis and design under uncertainty.

In this chapter, the fundamental mathematics of probability theory will be discussed. The discussions will focus on the basic concepts. The concepts will be explained and interpreted from an engineering perspective. The applications of probability theory in system reliability will also be introduced.

2.2 Set

A set is any collection of objects in which order has no significance, and multiplicity is generally also ignored. For example, integers from 1 through 10 form a set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}; the standard diameters of a transmission shaft of a manufacturing company form another set {1.5, 2.0, 2.5, 2.0, 3.5, 4.0} cm. A capital letter is used to represent a set.

Members of a set are referred to as elements. The notation $a \in \mathbf{A}$ is used to denote that *a* is an element of set **A**.

Order of elements in a set has no significance. Therefore, set {1, 2, 3} is equivalent to set {3, 1, 2}. Multiplicity is also generally ignored. For example, set {1, 1, 2, 3} is equivalent to set {1, 2, 3}.

The set of all elements under consideration is called a *universal set*, and the set containing no element is called an *empty* or *null set*. A empty set **A** is usually denoted by $\mathbf{A} = \varnothing$.

Operations on sets will be discussed below.

2.3 Basic Concepts

Experiment (**E**): An experiment is a well-defined action that results in a number of outcomes. For example, the rolling of a six-sided die is an example of experiment.

Outcome (**O**): An **outcome** is the result of a single trial of an experiment. For example, the outcomes rolling a six-sided die are 1, 2, 3, 4, 5, and 6.

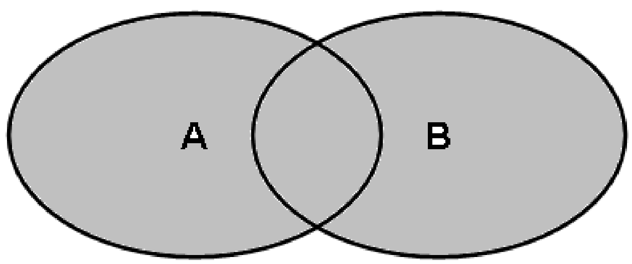
Universal space (**U**): The universal space is defined as the set of all possible outcomes of an experiment. For Example: The sample space of rolling a six-sided die is $\mathbf{U} = \{1, 2, 3, 4, 5, 6\}$.

A universal space is also called a *sample space*.

Event (**E**): An event is a collection of outcomes. An event is expressed by a set. For example, an event of rolling 1, 2, 3, or 6 is expressed by $\mathbf{A} = \{1, 2, 3, 6\}$, and an event of rolling 2, 3, 5, or 6 is expressed by $\mathbf{B} = \{2, 3, 5, 6\}$.

Union of two events **A** and **B** ($\mathbf{A} \cup \mathbf{B}$): The union of two events **A** and **B** is the set of outcomes that belong to **A** or **B** or both. For example, $\mathbf{A} \cup \mathbf{B} = \{1, 2, 3, 6\} \cup \{2, 3, 5, 6\} = \{1, 2, 3, 5, 6\}$.

The union of two sets **A** and **B** is illustrated in Fig. 2.1. Any element in **A** or **B** is the element of the union $\mathbf{A} \cup \mathbf{B}$.



A∪B

Figure 2.1 Union of Two Events

Intersection of two events **A** and **B** ($\mathbf{A} \cap \mathbf{B}$): The intersection of two events **A** and **B** is the set of outcomes that belong to both **A** and **B**.
For example, $\mathbf{A} \cap \mathbf{B} = \{1, 2, 3, 6\} \cap \{2, 3, 5, 6\} = \{2, 3, 6\}$.

The intersection of **A** and **B** is illustrated in Fig. 2.2. **T**he element that belongs to both **A** and **B** is the element of the intersection $\mathbf{A} \cap \mathbf{B}$.

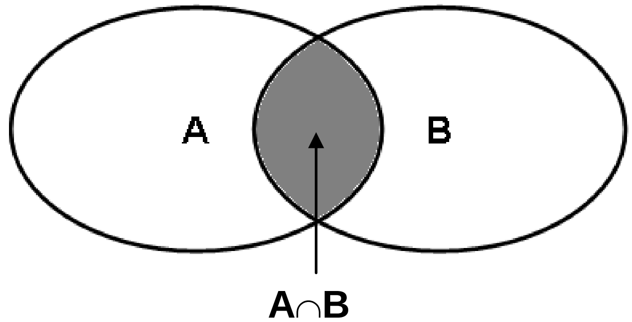


Figure 2.2 Union of Two Events

Complement of event **A** ($\overline{\mathbf{A}}$): A complement of an event **A** contains all outcomes of the universal set, **U**, that do not belong to **A**.

For example, $\overline{\mathbf{A}} = \{\overline{1,2,3,6}\} = \{4,5\}$.

The complement of event **A** is demonstrated in Fig. 2.3. All the elements that do not belong to **A** are the elements of the complement of **A**. In the figure, **U** is the universal set.

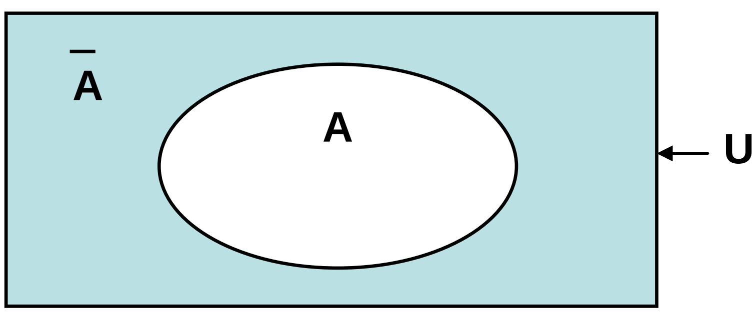


Figure 2.3 Complement of Event A

Null event ($\varnothing$): A null event is an empty set, and it has no outcome.

2.4 Probability

A probability refers to the likelihood of the occurrence of an event. The probability of an event **C** is the number of ways event **C** can occur divided by the total number of possible outcomes.

$$P(\mathbf{C}) = \frac{\text{The number of ways event C can occur}}{\text{The total number of possible outcomes}} \tag{2.1}$$

Eq. 2.1 indicates that a probability is a numerical measure of the likelihood of an event relative to a set of alternative events or the universal space.

Example 2.1

Suppose that event **C** is rolling an even number in the die tossing example; its probability is calculated as

$$P(\mathbf{C}) = \frac{\text{The number of ways } \mathbf{C} \text{ can occur}}{\text{The total number of possible outcomes}} = \frac{3}{6} = 0.5$$

A probability can be estimated by observations as shown in the following example.

Example 2.2

If the status of 1000 machines are monitored for three months and only two machines failed, then the probability of failure at the end of the third month is estimated by

$$P(\mathbf{A}) = \frac{\text{The number of machines failed}}{\text{The total number of machines}} = \frac{2}{1000} = 0.05$$

where **A** is the event of machine failure.

2.5 Probability Properties, Theorems and Axioms

The probability of an event **A** has the following properties:

1) $0 \le P(\mathbf{A}) \le 1$ (2.2)
2) $P(\mathbf{A}) = 1 - P(\overline{\mathbf{A}})$ (2.3)
3) $P(\varnothing) = 0$ (2.4)
4) $P(\mathbf{S}) = 1$ (2.5)

As indicated in Eqs. 2.4 and 2.5, when an event is certain to occur, it has a probability of 1; when it is impossible to occur, it has a probability of 0.

It is shown in Fig. 2.1 that the probability of the union of two events **A** and **B** is

$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \cap \mathbf{B}) \tag{2.6}$$

Similarly, using Eq. 2.6, the probability of the union of three events **A**, **B** and **C** is given by

$$P(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) = P(\mathbf{A}) + P(\mathbf{B}) + P(\mathbf{C}) - P(\mathbf{A} \cap \mathbf{B}) - P(\mathbf{A} \cap \mathbf{C}) - P(\mathbf{B} \cap \mathbf{C}) + P(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}) \quad (2.7)$$

2.6 Mutually Exclusive Events

Two events **A** and **B** are said to be mutually exclusive if they cannot occur simultaneously, i.e. $\mathbf{A} \cap \mathbf{B} = \varnothing$. In such a case, because the probability of the intersection of these events $\mathbf{A} \cap \mathbf{B}$ is zero, the expression for the union of these two events reduces to the following equation.

$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) \quad (2.8)$$

The mutually exclusive events **A** and **B** are demonstrated in Fig. 2.4. It is seen that set **A** does not overlap with set **B**.

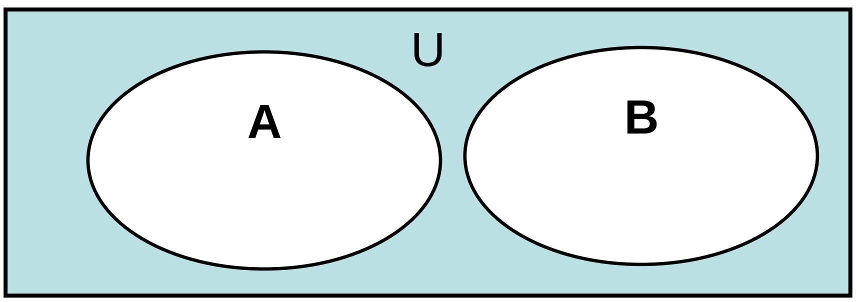


Figure 2.4 Mutually Exclusive Events

Eq. 2.8 can be generalized to *n* events. If events $\mathbf{A}_i \ (i = 1, 2, \cdots, n)$ are mutually exclusive, then

$$P(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \cdots \cup \mathbf{A}_n) = P(\mathbf{A}_1) + P(\mathbf{A}_2) + \cdots P(\mathbf{A}_n) \quad (2.9)$$

2.7 Conditional Probability

The conditional probability of two events **A** and **B** is defined as the probability of one of the events occurring knowing that the other event has already occurred. The expression below denotes the probability of **A** occurring given that **B** has already occurred; namely,

$P(\mathbf{A} | \mathbf{B})$ = the probability of **A** assuming the occurrence of **B**.

The conditional probability $P(\mathbf{A} | \mathbf{B})$ can be calculated by

$$P(\mathbf{A} | \mathbf{B}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{B})} \quad (2.10)$$

In what follows, we will use $\mathbf{AB}$ to denote $\mathbf{A} \cap \mathbf{B}$ for brevity.

Eq. 2.10 can be rewritten as

$$P(\mathbf{AB}) = P(\mathbf{A}|\mathbf{B})P(\mathbf{B}) = P(\mathbf{B}|\mathbf{A})P(\mathbf{A}) \tag{2.11}$$

Example 2.3

In a factory, 20%, 40%, and 40% of the components are manufactured by product lines 1, 2, and 3, respectively. The probabilities of having defective products from lines 1, 2, and 3 are 0.01, 0.015, and 0.02, respectively. What are the probabilities of defective products from each line?

Let the events be

$\mathbf{L}_1$ = Component produced from line 1
$\mathbf{L}_2$ = Component produced from line 2
$\mathbf{L}_3$ = Component produced from line 3
$\mathbf{D}$ = Defective component

The conditional probabilities are given by

$$P(\mathbf{D}|\mathbf{L}_1) = 0.01$$

$$P(\mathbf{D}|\mathbf{L}_2) = 0.015$$

$$P(\mathbf{D}|\mathbf{L}_3) = 0.02$$

The probabilities of defective products from each line are calculated with Eq. 2.12.

$$P(\mathbf{DL}_1) = P(\mathbf{D}|\mathbf{L}_1)P(\mathbf{L}_1) = 0.01 \times 0.2 = 0.002$$

$$P(\mathbf{DL}_2) = P(\mathbf{D}|\mathbf{L}_2)P(\mathbf{L}_2) = 0.015 \times 0.4 = 0.006$$

$$P(\mathbf{DL}_3) = P(\mathbf{D}|\mathbf{L}_3)P(\mathbf{L}_3) = 0.02 \times 0.4 = 0.008$$

2.8 Total Probability and Bayes' Theorem

If events $\mathbf{A}_i$ $(i = 1, 2, \cdots, n)$ are mutually exclusive and $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \cdots \cup \mathbf{A}_n = \mathbf{U}$; namely, the universal $\mathbf{U}$ is partitioned into n subsets $\mathbf{A}_i$ (Fig. 2.5). If $P(\mathbf{A}_i) \neq 0$, then for any event $\mathbf{B}$ (depicted by the ellipse in the figure) is

$$\mathbf{B} = (\mathbf{A}_1\mathbf{B}) \cup (\mathbf{A}_2\mathbf{B}) \cup \cdots \cup (\mathbf{A}_n\mathbf{B}) = \bigcup_{i=1}^{n} \mathbf{A}_i\mathbf{B} \ . \tag{2.12}$$

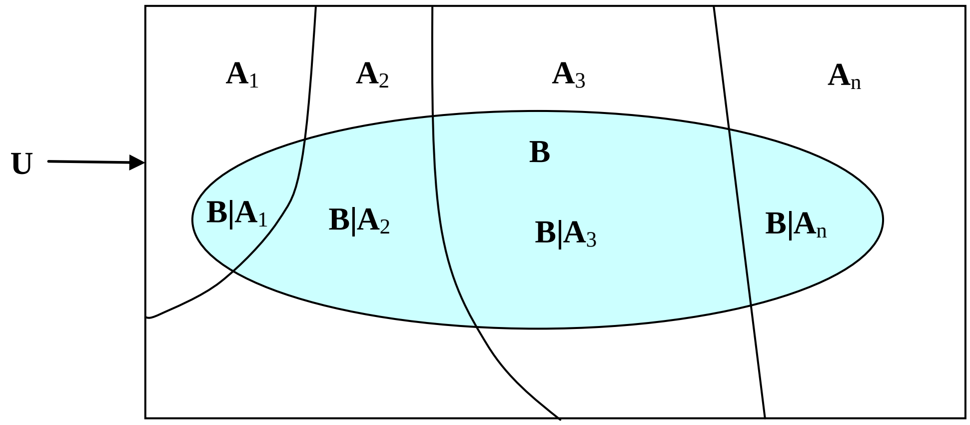


Figure 2.5 Partitioning the Sample Space S

Because events $\mathbf{A}_i\mathbf{B}$ are mutually exclusive (see Fig. 2.5), from Eq. 2.9,

$$P(\mathbf{B}) = P\left(\bigcup_{i=1}^{n} \mathbf{A}_i\mathbf{B}\right) = \sum_{i=1}^{n} P\left(\mathbf{A}_i\mathbf{B}\right) \tag{2.13}$$

Using Eq. 2.11, we have

$$P(\mathbf{B}) = \sum_{i=1}^{n} P\left(\mathbf{A}_i\right) P\left(\mathbf{B} \middle| \mathbf{A}_i\right) \tag{2.14}$$

Eq. 2.14 is called the *theorem of total probability*.

The theorem of total probability provides a way to calculate the probability of event **B** that cannot be determined directly; the occurrence of **B** depends on the occurrence of other events, such as $\mathbf{A}_1$, $\mathbf{A}_2$, …, $\mathbf{A}_n$.

On occasion, we are interested in knowing the conditional reverse probability $P\left(\mathbf{A}_i \middle| \mathbf{B}\right)$, the probability of a particular event $\mathbf{A}_i$ given the occurrence of **B**. Based on the same conditions of total probability theorem, the Bayes' Theorem is derived below.

According to Eq. 2.11

$$P(\mathbf{A}_i|\mathbf{B})P(\mathbf{B}) = P(\mathbf{B}|\mathbf{A}_i)P(\mathbf{A}_i)$$

we have

$$P\left(\mathbf{A}_i \middle| \mathbf{B}\right) = \frac{P\left(\mathbf{A}_i\right) P\left(\mathbf{B} \middle| \mathbf{A}_i\right)}{P\left(\mathbf{B}\right)}$$

Substituting Eq. 2.14 into the above equation yields the Bayes' Theorem

$$P\left(\mathbf{A}_i \middle| \mathbf{B}\right) = \frac{P\left(\mathbf{A}_i\right) P\left(\mathbf{B} \middle| \mathbf{A}_i\right)}{\sum_{j=1}^{n} P\left(\mathbf{A}_j\right) P\left(\mathbf{B} \middle| \mathbf{A}_j\right)} \tag{2.15}$$

The quantity $P\left(\mathbf{A}_i \middle| \mathbf{B}\right)$ is called the posterior probability, and $\left(\mathbf{B} \middle| \mathbf{A}_i\right)$ is called prior probability. The utility of Bayes' Theorem is demonstrated by the following example.

Example 2.4

For the same example in Section 2.7, if a component is found to be defective, we would be interested in the probability that the component was manufactured from a specific product line. For example, if a component is detected defective, what is the probability that the component was from line 2?

The probability of defective products can be calculated by the total probability theorem (Eq. 2.14) as

$$P(\mathbf{D}) = P\left(\mathbf{D} \middle| \mathbf{L}_1\right) P\left(\mathbf{L}_1\right) + P\left(\mathbf{D} \middle| \mathbf{L}_2\right) P\left(\mathbf{L}_2\right) + P\left(\mathbf{D} \middle| \mathbf{L}_3\right) P\left(\mathbf{L}_3\right)$$
$$= 0.01 \times 0.2 + 0.015 \times 0.4 + 0.02 \times 0.4 = 0.016$$

According to the Bayes' Theorem in Eq. 2.15, the posterior probability is calculated as

$$P(\mathbf{L}_2 | \mathbf{D}) = \frac{P\left(\mathbf{D} \middle| \mathbf{L}_2\right) P\left(\mathbf{L}_2\right)}{P\left(\mathbf{D} \middle| \mathbf{L}_1\right) P\left(\mathbf{L}_1\right) + P\left(\mathbf{D} \middle| \mathbf{L}_2\right) P\left(\mathbf{L}_2\right) + P\left(\mathbf{D} \middle| \mathbf{L}_3\right) P\left(\mathbf{L}_3\right)}$$
$$= \frac{0.015 \times 0.4}{0.016} = 0.375$$

2.9 Independent Events

If knowing event **B** gives no information about event **A**, or the occurrence of **B** does not affect that of **A**, then both of the events are said to be independent. The conditional probability expression reduces to

$$P(\mathbf{A}|\mathbf{B}) = P(\mathbf{A}) \tag{2.16}$$

From the definition of conditional probability, Eq. 2.10 can be rewritten as

$$P(\mathbf{A}) = P(\mathbf{A}|\mathbf{B}) = \frac{P(\mathbf{AB})}{P(\mathbf{B})} \tag{2.17}$$

This leads to the expression

$$P(\mathbf{AB}) = P(\mathbf{A})P(\mathbf{B}) \tag{2.18}$$

Similarly, if a group of *n* events $\mathbf{A}_i$ are mutually independent, then

$$P(\bigcap_{i=1}^{n} \mathbf{A}) = \prod_{i=1}^{n} P(\mathbf{A}_i) \tag{2.19}$$

2.10 Applications of Probability Theory in System Reliability

Reliability is the ability of a device or system to perform its required function under stated conditions for a specified period of time. Reliability is quantified by the probability that a device or a system performs its required function for a specified interval under stated conditions. In many engineering applications, the reliability of a system is difficult to estimate. But the reliability of its components is relatively easy to estimate. Knowing the component reliability and the logical relationship between the system and its components, it is possible to calculate the system reliability. Next, we will see how to calculate the system reliability for three typical types of systems by using probability theory we have discussed.

2.10.1 The reliability of a series system

A series system consisted of two components is shown in Fig. 2.6. The failure of either of the two components can result in the failure of the entire system. This type of system is also referred to as a weakest link system.

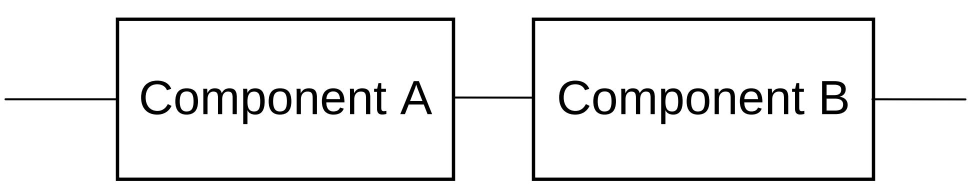


Figure 2.6 A Series System

Example 2.5

Suppose the following information is obtained from the field data.

- Event **A**: failure of Component A, P(**A**)=0.01
- Event $\overline{\mathbf{A}}$: good condition of Component A, $P(\overline{\mathbf{A}}) = 1 - P(\mathbf{A}) = 0.99$
- Event **B**: failure of Component B, P(**B**)=0.02
- Event $\overline{\mathbf{B}}$: good condition of Component B, $P(\overline{\mathbf{B}}) = 1 - P(\mathbf{B}) = 0.98$
- Events **A** and **B** are independent: the state of one component does not affect that of the other.

The reliability is defined as the probability of good condition. Knowing the reliability of individual components, we are interested in knowing the system reliability, which is the probability of system in good condition.

Since a system failure occurs when either component fails, the system failure event **C** is the union of events **A** and **B**, and therefore the probability of system failure is

$$P(\mathbf{C}) = P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{AB}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A})P(\mathbf{B})$$
$$= 0.01 + 0.02 - 0.01 \times 0.02 = 0.0298$$

The event of system in good condition is the complement of system failure event **C**, and therefore the system reliability is given by

$$R = 1 - P(\overline{\mathbf{C}}) = 1 - 0.0298 = 0.9702$$

The problem can also be solved in another way. Since if only both components are in good condition, the system is in good condition, the event of system in good condition is the intersection of events of individual components in good condition. Therefore, the reliability is calculated by

$$R = P(\overline{\mathbf{A}}\overline{\mathbf{B}}) = P(\overline{\mathbf{A}})P(\overline{\mathbf{B}}) = 0.99 \times 0.98 = 0.9702$$

From this example, it is noted that the system reliability of a series system is less than the minimum reliability of individual components.

The above equation can also be written as

$$R = R_1 R_2 \tag{2.20}$$

where $R_1 = P(\overline{\mathbf{A}})$ and $R_2 = P(\overline{\mathbf{B}})$, each of which is the reliability of components 1 and 2, respectively.

The conclusion can be generalized to a series system of n components as shown in Fig. 2.7. If the reliability of component i is R_i and the failure events of all the components are independent, the system reliability R is then given by

$$R = \prod_{i=1}^{n} R_i \tag{2.21}$$

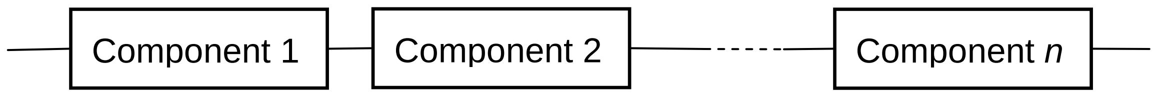


Figure 2.7 A General Series System

2.10.2 The reliability of a parallel system

A parallel system consisting of two components is shown in Fig. 2.8. Only if both of the components fail, the system fails. This type of system is also called a redundant system.

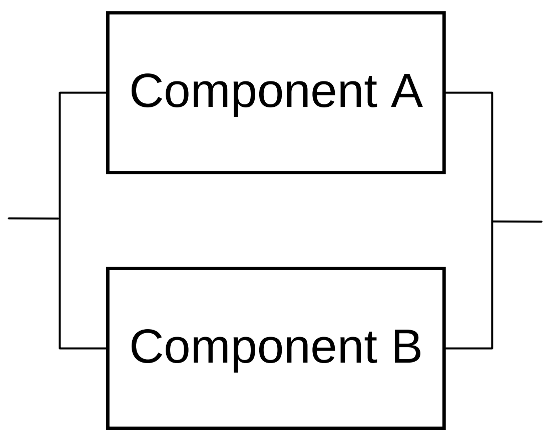


Figure 2.8 A Parallel System

Example 2.6

If we have the same information as in Example 2.4, what is the system reliability?

Since the event of system failure is the intersection of events that component 1 fails and component 2 fails, the probability of system failure is calculated by

$$P(\mathbf{C}) = P(\mathbf{AB}) = P(\mathbf{A})P(\mathbf{B}) = 0.01 \times 0.02 = 0.002$$

Then the system reliability is given by

$$R = P(\overline{\mathbf{C}}) = 1 - P(\mathbf{C}) = 1 - 0.002 = 0.998$$

Obviously, the system reliability is greater than the reliability of individual components.

The above equation can also be written as

$$R = 1 - [1 - P(\overline{\mathbf{A}})][1 - P(\overline{\mathbf{B}})] = 1 - (1 - R_1)(1 - R_2) \tag{2.22}$$

The conclusion can be generalized to a parallel system of *n* components. If the reliability of component *i* is R_i and the failure events of all the components are independent, the system reliability *R* is then given by

$$R = 1 - \prod_{i=1}^{n} (1 - R_i) \tag{2.23}$$

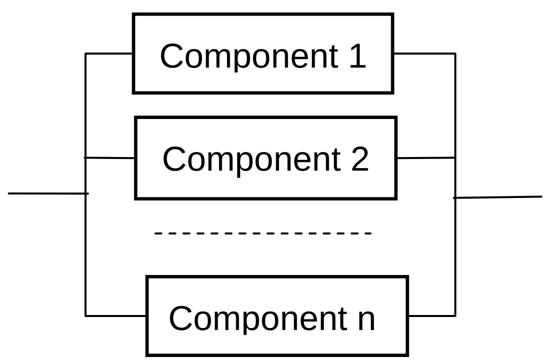


Figure 2.9 A General Parallel System

2.10.3 The reliability of a system with mixed series and parallel structure

A general system is mixed with components in series and parallel. Fig. 2.10 depicts such a mixed system.

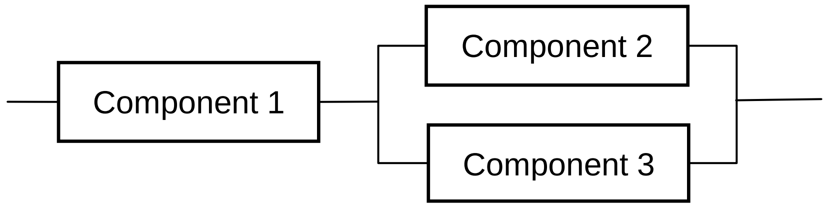


Figure 2.10 A Mixed System

The reliability of a mixed system can be easily derived from the above equations for a series system and parallel system. For example, for the mixed system shown in Fig. 2.10, Components 2 and 3 form a parallel subsystem, and the subsystem and component 1 form a series system. The system reliability can be calculated as follows.

From Eq. 2.22, the reliability of the parallel subsystem is calculated by

$$R_{2\text{-}3} = 1 - (1 - R_2)(1 - R_3)$$

From Eq. 2.20, the system reliability is then given by

$$R = R_1 R_{2\text{-}3} = R_1 \left[1 - (1 - R_2)(1 - R_3)\right]$$