Chapter 9

Other Uncertainty Analysis Methods

9.1 Introduction

In Chapter 7, we have discussed reliability analysis methods FORM and SORM. Both of the methods can provide reasonably accurate reliability analysis results. But they may be inefficient when the number of random variables is large and when the derivative of a performance function is evaluated numerically (for example, through the finite difference method). The reason is that both of the methods require searching the MPP and that this search process involves a number of deterministic analyses. Furthermore, the distributions of the input random variables have to be known. In some engineering applications, a precise distribution of a random variable may not be available because of limited information. For example, only a small sample size of a random variable is available, only the mean and standard deviation can be obtained, or only the interval where a random variable resides is known. For these situations, one may consider using the moment matching method or the worst case analysis method that we will discuss below. If the evaluation of the performance function is computationally expensive, a simplified model must be created to replace the original performance function for the uncertainty analysis purpose. Response surface method (RSM) is one of such methods and will also be discussed in this chapter.

9.2 Moment Matching Method

If the first two moments (mean and standard deviation) of a random variable are known, the moment matching method can be used to estimate the mean and standard deviation of a performance function. Then the mean and standard deviation of the performance function may be used to estimate the probability of failure.

Assume that the random variables $\mathbf{X}=\left(X_1, X_2, \cdots, X_n\right)$ are mutually independent and that the means and standard deviations of $\mathbf{X}=\left(X_1, X_2, \cdots, X_n\right)$ are $\boldsymbol{\mu}=\left(m_1, m_2, \cdots, m_n\right)$ and $\mathbf{s}=\left(s_1, s_2, \cdots, s_n\right)$, respectively. The first order Taylor expansion of the performance function $g(\mathbf{X})=g\left(X_1, X_2, \cdots, X_n\right)$ at the means $\boldsymbol{\mu}=\left(m_1, m_2, \cdots, m_n\right)$ provides the following linearization

$$g(\mathbf{X}) \approx L(\mathbf{X})= g(\boldsymbol{\mu})+\sum_{i=1}^{n}\left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{\boldsymbol{\mu}}(X_i-m_i)=g(\boldsymbol{\mu})+\nabla g(\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^T, \tag{9.1}$$

where $\nabla g(\boldsymbol{\mu})$ is the gradient of $g(\mathbf{U})$ at $\boldsymbol{\mu}=\left(m_1, m_2, \cdots, m_n\right)$. $\nabla g(\boldsymbol{\mu})$ is given by

$$\nabla g(\boldsymbol{\mu}) = \left(\frac{\partial g(\mathbf{X})}{\partial X_1}, \frac{\partial g(\mathbf{X})}{\partial X_2}, \cdots, \frac{\partial g(\mathbf{X})}{\partial X_n} \right)\Bigg|_{\boldsymbol{\mu}} \tag{9.2}$$

Then, the mean of $g(\mathbf{X})$ is approximated by the mean of the linearized function $L(\mathbf{X})$ and is given by

$$m_g \approx g(\boldsymbol{\mu}). \tag{9.3}$$

The standard deviation of $g(\mathbf{X})$ is given by

$$s_g \approx \sqrt{\sum_{i=1}^{n} \left[\frac{\partial g(\mathbf{X})}{\partial X_i}\bigg|_{\boldsymbol{\mu}} \right]^2 s_i^2}. \tag{9.4}$$

If the input random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ are assumed to be normally distributed, the linearized performance function $L(\mathbf{X})$ is also normally distributed since $L(\mathbf{X})$ is a linear combination of normal variables. The probability of failure is therefore computed by

$$p_f \approx P\{L(\mathbf{X}) < 0\} = \Phi\left(\frac{-m_g}{s_g}\right) = \Phi\left(-g(\boldsymbol{\mu}) \Bigg/ \sqrt{\sum_{i=1}^{n} \left[\frac{\partial g(\mathbf{X})}{\partial X_i}\bigg|_{\boldsymbol{\mu}} \right]^2 s_i^2} \right). \tag{9.5}$$

The moment matching method is also called the first order and the second moment (FOSM) method since it involves the first order derivative and second moment. Two approximations are used in the moment matching method – the distributions of random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ are assumed normally distributed, and the performance function $g(\mathbf{X})$ is linearized at the means of $\mathbf{X} = (X_1, X_2, \cdots, X_n)$.

FOSM does not need the MPP search and it is more efficient than both FORM and SORM. If the derivative of $g(\mathbf{X})$ is evaluated numerically by the finite difference method, the total number of function evaluation is n + 1. n evaluations are for the derivative calculation, and one evaluation is for the calculation of the performance at the means of $\mathbf{X} = (X_1, X_2, \cdots, X_n)$. In many cases, FOSM can provide satisfactorily accurate results. Because of its easiness and efficiency, FOSM has been widely used, especially in probabilistic mechanical component design and robust design.

However, FOSM is not accurate as FORM or SORM in general. As shown in Fig. 9.1, the probability of failure p_f is estimated by the probability integration over the hatched region, which is on the upper right side of the straight line $g(\boldsymbol{\mu})+\nabla g(\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^T=0$. The actual probability integration region is the shaded region that is on the upper right side of the curve $g(\mathbf{X})=0$. Because of the approximations, FORM may result in a significant error for highly nonlinear functions with random variables that follow nonnormal distributions.

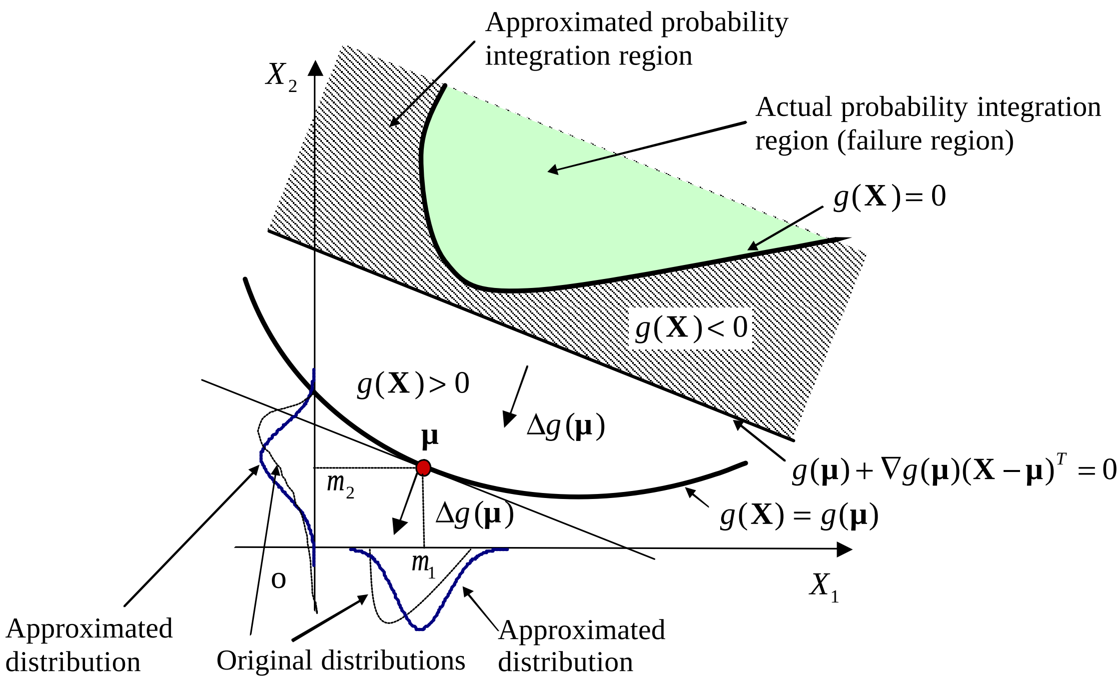

Figure 9.1 Approximations in Moment Matching Method

Example 9.1 – reliability analysis of the cantilever beam

Use FORM to solve Example 7.2.

The performance function is given by

$$g=D_0-\frac{4L^3}{Ewt}\sqrt{\left(\frac{P_y}{t^2}\right)^2+\left(\frac{P_x}{w^2}\right)^2},$$

where $P_x \sim N(500,100)lb$ and $P_y \sim N(1000,100)lb$, and the other parameters in the above equation are constants (see Example 7.2). The gradient of the performance function is given by

$$\nabla g(\mathbf{X}) = \left(\frac{\partial g}{\partial P_x}, \frac{\partial g}{\partial P_y} \right) = -\frac{4L^3}{Ewt} \left(\frac{P_x}{w^4 \sqrt{\left(\frac{P_y}{t^2}\right)^2 + \left(\frac{P_x}{w^2}\right)^2}}, \ \frac{P_y}{t^4 \sqrt{\left(\frac{P_y}{t^2}\right)^2 + \left(\frac{P_x}{w^2}\right)^2}} \right).$$

The mean of $g(\mathbf{X})$ is computed by

$$m_g \approx g(\boldsymbol{\mu}) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{m_{P_y}}{t^2}\right)^2 + \left(\frac{m_{P_x}}{w^2}\right)^2} = 3 - \frac{4\times 100^3}{30\times 10^6 \times 2 \times 4}\sqrt{\left(\frac{1000}{4^2}\right)^2 + \left(\frac{500}{2^2}\right)^2}$$
$$= 0.6708$$

and the standard deviation of $g(\mathbf{X})$ is computed by

$$s_g = \sqrt{\left(\left.\frac{\partial g}{\partial P_x}\right|_{m_{P_x}, m_{P_y}} \right)^2 s_{P_x}^2 + \left(\left.\frac{\partial g}{\partial P_y}\right|_{m_{P_x}, m_{P_y}} \right)^2 s_{P_y}^2}$$
$$= \frac{4L^3}{Ewt}\sqrt{\left[m_{P_x} \Big/ w^4 \sqrt{\left(\frac{m_{P_y}}{t^2}\right)^2 + \left(\frac{m_{P_x}}{w^2}\right)^2} \right]^2 s_{P_x}^2 + \left[m_y \Big/ t^4 \sqrt{\left(\frac{m_{P_y}}{t^2}\right)^2 + \left(\frac{m_{P_x}}{w^2}\right)^2} \right]^2 s_{P_y}^2} = 0.3734$$

The probability of failure is then calculated by

$$p_f \approx P\{L(\mathbf{U}) < 0\} = \Phi(\frac{-m_g}{s_g}) = \Phi\left(\frac{-0.6708}{0.3734}\right) = 0.03622\,.$$

The solution of the probability of failure from Monte Carlo Simulation with 10^5 simulations is $p_f = 0.04143$ (see Example 8.2), and the solution from FORM is $p_f = 0.040541$ (see Example 7.2). If we consider the Monte Carlo Simulation solution as the accurate solution, the solution from FOSM is not as accurate as that of FORM. However, FOSM is much more efficient than FORM since the former does not need to search the MPP.

As the example demonstrates, FOSM only requires the function value and its gradient at the means of random variables. It is efficient and is therefore widely used in many engineering applications, especially in robust design. It should be noted that one potential problem of FOSM is that if a performance function expressed by another equivalent formulation, the solution of the probability of failure may vary. For example, if the above performance function is rewritten with an equivalent formulation

$$g = D_0^2 - \left(\frac{4L^3}{Ewt}\right)^2 \left[\left(\frac{P_y}{t^2}\right)^2 + \left(\frac{P_x}{w^2}\right)^2\right],$$

the probability of failure from FOSM will become $p_f = 0.0205$, which is quite different from the solution based on the original formulation.

FOSM is also commonly applied in probabilistic mechanical component design. The following example demonstrates such an application.

Example 9.2 – probabilistic shaft design

A shaft subjected to an axial forces Q is shown in Fig. 9.2. The mean and standard deviation of Q are $m_Q = 40$ kN and $s_Q = 1.2$ kN, respectively. The mean and standard deviation of the yield strength S_y are $m_{S_y} = 667$ MPa and $s_{S_y} = 25.3$ MPa, respectively. The standard deviation of the shaft radius r is $s_r = 10^{-3}$ m. Determine the mean of the radius m_r such that the reliability of the component is $R = 0.999$.

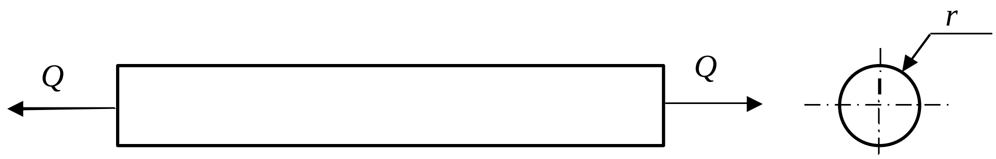

Figure 9.2 A Shaft Subjected to Axial Forces

In this design problem, there is only one design variable, which is the radius of the shaft. For comparison, the conventional (deterministic) design is also given below.

Deterministic design

In the deterministic design, the means (nominal values) are used. To make sure that the design is safe enough, a large safety factor $S_F = 3$ is chosen. The normal stress of the shaft is calculated by

$$S = \frac{Q}{p r^2}.$$

From the definition of the safety factor S_F

$$S_F = \frac{S_y}{S} = \frac{S_y p r^2}{Q},$$

The radius is then determined by

$$r = \sqrt{\frac{S_F Q}{p\, S_y}} = \sqrt{\frac{3 \times 40 \times 10^3}{p \times 667 \times 10^6}} = 7.6 \times 10^{-3} m = 7.6 mm\,.$$

Probabilistic design

The performance function is defined as

$$g = S_y - S = S_y - \frac{Q}{p\, r^2}\,.$$

The mean of g is given by

$$m_g = m_{S_y} - \frac{m_Q}{p m_r^2}\,.$$

The standard deviation of g is calculated by

$$s_g = \sqrt{\left(\frac{\partial g}{\partial Q}\right)^2 s_Q^2 + \left(\frac{\partial g}{\partial r}\right)^2 s_r^2 + \left(\frac{\partial g}{\partial S_y}\right)^2 s_{S_y}^2}$$

$$= \sqrt{\left(\frac{-1}{p m_r^2}\right)^2 s_Q^2 + \left(\frac{2 m_Q}{p m_r^3}\right)^2 s_r^2 + (1)^2 s_{S_y}^2} = \sqrt{\left(\frac{1}{p m_r^2}\right)^2 s_Q^2 + 4\left(\frac{m_Q}{p m_r^3}\right)^2 s_r^2 + s_{S_y}^2}\,.$$

Then, the probability of failure is computed by

$$p_f = P(g < 0) = \Phi\left(\frac{-m_g}{s_g}\right) = 1 - R\,.$$

Therefore,

$$\frac{-m_g}{s_g} = \Phi^{-1}(1 - R)\,,$$

or,

$$m_g + \Phi^{-1}(1 - R) s_g = 0\,.$$

Substituting the equations of m_g and s_g into the above equation yields

$$m_{S_y} - \frac{m_Q}{p m_r^2} + \Phi^{-1}(1-R)\sqrt{\left(\frac{1}{p m_r^2}\right)^2 s_Q^2 + 4\left(\frac{m_Q}{p m_r^3}\right)^2 s_r^2 + s_{S_y}^2} = 0\,,$$

which is

$$667\times10^6 - \frac{40\times10^3}{p m_r^2} + \Phi^{-1}(1-0.999)\sqrt{\left(\frac{1}{p m_r^2}\right)^2 \left(1.2\times10^3\right)^2 + 4\left(\frac{40\times10^3}{p m_r^3}\right)^2 \left(1\times10^{-3}\right)^2 + \left(25.3\times10^6\right)^2} = 0$$

The solution to the above equation is $m_r = 6.22\times10^{-3}\ \text{m} = 6.22\ \text{mm}$.

Comparing the result from the probabilistic design and that of the deterministic design, it is seen that the latter is much more conservative because it requires a larger radius. With the required reliability of 0.999, the deterministic approach “over designed” the component and resulted in a higher reliability than required. (Interested readers may want to calculate the reliability for the deterministic design.) On the other hand, if a smaller factor of safety (e.g. 1.5) were used, the deterministic design would be risky since the reliability would be less than the required value.

As discussed previously, FOSM may not result in accurate reliability estimation. If higher accuracy is desired for the design, one should use the advanced probabilistic design methods that will be discussed in Part III in this book.

9.3 Worst Case Analysis

Some times the information about random variables may be limited. We only know the interval over which a random variable may reside. Some of the situations where the uncertainties are characterized with intervals are as follows.

(1) “Sometimes a quantity may not have been studied at all, and the only real information about it comes from theoretical constraints. Physical limits may be used to circumscribe possible ranges of quantities even when no empirical information about them is available.” [1]

(2) Condition monitoring calls for periodic inspection of components. A component may be in working condition at one inspection, but in failure condition at the next. The time to failure is therefore in a window of time between the last two inspections during which the component failed.

(3) A measurement from a device is associated with an interval based on the number of reported digits. For example, the value 9.32 may be associated with the interval [9.30, 9.35] where the two endpoints are the two closest landmark values.

(4) Design engineers often specify their design variables in the form of nominal values plus-or-minus tolerances. For a completely new design, it is hard to know how the design variables could be distributed over the tolerance ranges before physical deployment [2].

(5) Many engineering formulations have their application limits. Intervals are used to identify the choices of formulations. For example, if the ratio of thickness to internal diameter of a cylinder is between 1.1 and 1.2, the cylinder is considered as a thin-wall cylinder; if the ratio is greater than 1.2, it is considered as a thick-wall cylinder [2].

(6) Analysis engineers often estimate the analysis error by the percentage of the nominal analysis results. For example, for a particular application, a finite element analysis may be reported to have a 10% error [2].

For situations where only intervals of random variables are known, we can use the worst case analysis to find the interval of a performance function.

Assume that the interval for random variable X_i is $[a_i,\ b_i]$, then its average is given by

$$\overline{X}_i = \frac{a_i + b_i}{2}. \tag{9.7}$$

Obviously, the distance between the average and one of the endpoints is half of the range $b_i - a_i$, namely,

$$\Delta_i = b_i - \overline{X}_i = \overline{X}_i - a_i = \frac{b_i - a_i}{2}. \tag{9.8}$$

The procedure of worst case analysis is described as follows.

First, the performance function is linearized by the first order Taylor expansion at the average of the input random variables $\overline{\mathbf{X}} = \left(\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_n\right)$ by

$$g(\mathbf{X}) \approx g(\overline{\mathbf{X}}) + \sum_{i=1}^{n} \left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{\overline{\mathbf{X}}} (X_i - \overline{X}_i). \tag{9.9}$$

The average of the performance function is evaluated at the average $\overline{\mathbf{X}} = \left(\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_n\right)$, namely,

$$\overline{g}(\mathbf{X}) = g(\overline{\mathbf{X}}). \tag{9.10}$$

The deviation of the performance function from its average is then given by

$$\Delta g = g(\mathbf{X}) - g(\overline{\mathbf{X}}) = \sum_{i=1}^{n} \left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{\overline{\mathbf{X}}} (X_i - \overline{X}_i). \tag{9.11}$$

Because we are interested in the worst case, we take the absolution values of the derivatives in the above equation and use the maximum changes of random variables Δ_i. Then, the worst case difference (the maximum difference) of the performance function is given by

$$\Delta g = \sum_{i=1}^{n} \left| \frac{\partial g(\mathbf{X})}{\partial X_i} \bigg|_{\overline{\mathbf{X}}} \right| \Delta X_i = \frac{1}{2} \sum_{i=1}^{n} \left| \frac{\partial g(\mathbf{X})}{\partial X_i} \bigg|_{\overline{\mathbf{X}}} \right| (b_i - a_i) . \tag{9.12}$$

Therefore, the performance will vary in the follow range,

$$\left[\overline{g} - \Delta g,\ \overline{g} + \Delta g \right] = \left[g(\overline{\mathbf{X}}) - \sum_{i=1}^{n} \left| \frac{\partial g(\mathbf{X})}{\partial X_i} \bigg|_{\overline{\mathbf{X}}} \right| \Delta X_i ,\ g(\overline{\mathbf{X}}) + \sum_{i=1}^{n} \left| \frac{\partial g(\mathbf{X})}{\partial X_i} \bigg|_{\overline{\mathbf{X}}} \right| \Delta X_i \right]. \tag{9.13}$$

If the safe region is defined by $g(\mathbf{X}) > 0$, the worst case performance function will be $\overline{g} - \Delta g$, and this worst case value should be greater than 0, namely, $\overline{g} - \Delta g > 0$.

It should be noted that the above method for identifying the worst case of the performance function is an approximation. The approximation comes from using the first order Taylor expansion and taking the absolute value of the derivatives. Therefore, the solution from Eq. 9.12 has some error. To accurately identify the worst case value of a performance function, the maximum (or minimum) value of the performance has to be searched over the ranges of all the input variables. Optimization techniques can be used for this purpose. It should also be noted that the results from the worst case scenario may be too conservative.

Example 9.3 – worst case analysis of the cantilever beam

The same beam problem as in Example 9.1 is used to demonstrate the worst case analysis method. The tip displacement of the cantilever beam is given by

$$g = \frac{4L^3}{Ewt} \sqrt{\left(\frac{P_y}{t^2} \right)^2 + \left(\frac{P_x}{w^2} \right)^2} ,$$

where the external forces P_x and P_y are known in the intervals $[400, 600]$ and $[800,\ 1200]$, respectively. Therefore, $\Delta P_x = 100$, and $\Delta P_y = 200$.

The averages of P_x and P_y are $\overline{P}_x = 500$ and $\overline{P}_y = 1000$, respectively. The average of g is computed by

$$\bar{g} = \frac{4L^3}{Ewt}\sqrt{\left(\frac{\bar{P}_y}{t^2}\right)^2 + \left(\frac{\bar{P}_x}{w^2}\right)^2} = \frac{4\times 100^3}{30\times 10^6\times 2\times 4}\sqrt{\left(\frac{1000}{4^2}\right)^2 + \left(\frac{500}{2^2}\right)^2} = 0.67076 \text{ in}$$

The gradient of g at the averages of P_x and P_y is given by

$$\nabla g = \frac{4L^3}{Ewt}\left[\frac{\bar{P}_x}{w^4\sqrt{\left(\frac{\bar{P}_y}{t^2}\right)^2 + \left(\frac{\bar{P}_x}{w^2}\right)^2}},\ \frac{\bar{P}_y}{t^4\sqrt{\left(\frac{\bar{P}_y}{t^2}\right)^2 + \left(\frac{\bar{P}_x}{w^2}\right)^2}}\right]$$

$$= \frac{4\times 100^3}{30\times 10^6\times 2\times 4}\left[\frac{500}{2^4\sqrt{\left(\frac{1000}{4^2}\right)^2 + \left(\frac{500}{2^2}\right)^2}},\ \frac{1000}{4^4\sqrt{\left(\frac{1000}{4^2}\right)^2 + \left(\frac{500}{2^2}\right)^2}}\right]$$

$$= \left(\text{-0.0037268, -0.00023292}\right).$$

Then,

$$\Delta g = \left|\left.\frac{\partial g}{\partial P_x}\right|_{\bar{\mathrm{x}}}\right|\Delta P_x + \left|\left.\frac{\partial g}{\partial P_y}\right|_{\bar{\mathrm{x}}}\right|\Delta P_y = 0.0037268\times 100 + 0.00023292\times 200 = 0.41926 \text{ in}$$

Thus, the range of the performance function is given by

$$[g_{\min}, g_{\max}] = \left[\bar{g} - \Delta g, \bar{g} + \Delta g\right] = [0.2515, 1.09] \text{ in}.$$

The largest (worst case) displacement is

$$g_{worst} = \bar{g} + \Delta g = 1.09 \text{ in}.$$

If the allowable tip displacement is 1.5, the design will be considered safe. However, if the allowable tip displacement is 1.0, the design will be considered as a failure; in this case, the design has to be modified.

Example 9.4 – shaft design by worst case analysis

The shaft design problem in Example 9.2 is solved again by the worst case analysis. The ranges of the random variables are set to 3 standard deviations, namely, $\Delta r = 3s_r = 3\times 10^{-3}$ m , $\Delta Q = 3s_Q = 3\times 1.2 = 3.6$ kN , $\Delta_{S_y} = 3s_{S_y} = 3\times 25.3 = 75.9$ MPa .

The averages of the uncertain variables are $\bar{r}$, $\bar{Q} = 40$ kN, and $\bar{S}_y = 66.7$ MPa. The average radius $\bar{r}$ is to be determined.

The average performance function is given by

$$\bar{g} = \bar{S}_y - \frac{\bar{Q}}{p\bar{r}^2}.$$

The range of g is given by

$$\Delta_g = \left|\frac{\partial g}{\partial Q}\right|\Delta_Q + \left|\frac{\partial g}{\partial r}\right|\Delta r + \left|\frac{\partial g}{\partial S_y}\right|\Delta_{S_y} = \left|\frac{1}{pm_r^2}\right|\Delta_Q + \left|\frac{2m_Q}{pm_r^3}\right|\Delta r + \Delta_{S_y}.$$

The worst case g is $\bar{g} - \Delta g$ and should be greater than zero; therefore,

$$\bar{g} = \left(\bar{S}_y - \frac{\bar{Q}}{p\bar{r}^2}\right) - \left(\left|\frac{1}{pm_r^2}\right|\Delta_Q + \left|\frac{2m_Q}{pm_r^3}\right|\Delta r + \Delta_{S_y}\right) = 0,$$

which yields the following eqaution

$$\left(667\times10^6 - \frac{40\times10^3}{p\bar{r}^2}\right) - \left(\frac{1}{p\bar{r}^2}3.6\times10^3 + \frac{2\times40\times10^3}{p\bar{r}^3}3\times10^{-3} + 75.9\times10^6\right) = 0.$$

Solving the above equation yields the design variable, $\bar{r} = 6.6$ mm, which is greater than the radius obtained from moment matching method. This indicates that the worst case analysis may result in a conservative design.

9.4 Response Surface Method

In many engineering applications, the evaluation of a performance function is computationally expensive. Uncertainty analysis needs a number of such evaluations. One solution to this problem is to create a surrogate model to replace the original expensive performance function. The evaluation of a surrogate model is much cheaper than that of the original performance function. The basic idea is to perform a number of experiments (numerically or physically) at different design points (or inputs), and then the performance function values and corresponding inputs are used to fit the simplified surrogate model. This process is called Design of Experiments (DOE), or more precisely, Computer Design of Experiments if the experiments are conducted numerically. Once a surrogate model is established, the uncertainty analysis methods such as Monte Carlo Simulation, FORM, and SORM can be applied for uncertainty analysis.

There are several tasks in DOE, including selecting the type of surrogate model, identifying design points where the experiments will be performed, and solving the unknown coefficients of the surrogate model. Generally, the functions which can accurate represent the original function and need a small number of experiments are favorable. Response Surface Model (RSM) is one of those functions. RSM is a polynomial type of function. Next we will use a simple example to discuss RSM with the following procedure.

The general procedure of RSM method is as follows.

Step 1: Determine the design (input) variables and response variables.
Step 2: Determine the design variable bounds.
Step 3: Plan the experiment, including the number of experiments, levels of design variables, and the type of response surface.
Step 4: Perform experiment to obtain the response variables at the design points determined in Step 3.
Step 5: Determine the unknown coefficients of the response surface model and perform other analyses such as sensitivity analysis.
Step 6: Use the response surface model for uncertainty analysis.

Next, a simple example will be used to demonstrate a 2-level full factorial design, where two points (levels) of each design variable and all the combinations of design variable levels are used. In this example, the response Y is calculated a some computer simulation program, which is very time consuming. To get a cheaper model of the response in terms of design variables, the DOE is performed. The design variables include two continuous variables X_1 and X_2, and a discrete variable X_3 that takes values of either A or B.

Step 1: Determine the design variables and response variables
The design variables are the dimensional variables X_1 and X_2, and the material type X_3. The first two dimensional design variables are considered continuous, and the last design variable X_3 is a discrete variable since it represents the type of materials.

Step 2: Determine the design variable bounds
The bounds of X_1 are $[X_{1\min}, X_{1\max}] = [-160,\ 170]mm$ and $[X_{2\min}, X_{2\max}] = [-20,\ 40]mm$, respectively. There are two material types available; therefore X_3 can be either Type A or Type B.

Step 3: Plan the experiment
Two levels for each design variable are considered for the experiment. For simplicity, all the design variables are transformed at the scale of [-1, +1], where -1 stands for the lower bound $X_{i\min}$, and +1 stands for the upper bound $X_{i\max}$. A continuous design variable $X_i^{'}$ at the scale of [-1, +1] is then expressed by

$$X_i^{'} = \frac{2(X_i - X_{i\min})}{X_{i\max} - X_{i\min}} - 1. \quad (9.14)$$

A two-level full factorial design is used for this problem. All the combinations (design points) of two levels of the design variables are considered, and the two levels are selected on the lower bounds and upper bounds of the design variables. The following table gives the design points. (The table is called *DOE matrix*). There are 8 experiments in the table. For X_3, -1 represents Type A material, and +1 represents Type B material.

Table 9.1 DOE Matrix

Experiment	$X_1^{'}$	$X_2^{'}$	$X_3^{'}$
1	-1	-1	-1
2	1	-1	-1
3	-1	1	-1
4	1	1	-1
5	-1	-1	1
6	1	-1	1
7	-1	1	1
8	1	1	1

A linear function of the stress Y is selected for the RSM, which is given by

$$Y = b_0^{'} + b_1^{'} X_0^{'} + b_2^{'} X_2^{'} + b_3^{'} X_3^{'}, \tag{9.15}$$

where $b_i^{'}$ $(i = 0,1,2,3)$ are unknown coefficients.

The DOE matrix is visualized in Fig. 9.3, where the circles represent the design points.

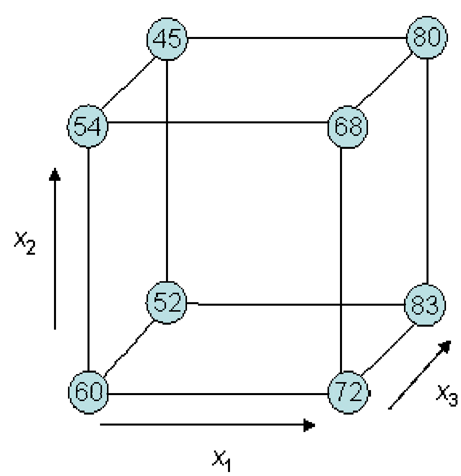

Figure 9.3 The visualization of the DOE Matrix

Since 8 values of the response Y are to be obtained from the 8 experiments, maximally, 8 unknown coefficients can be included in the RSM. Therefore, the following higher order polynomial with 8 unknown coefficients and interaction terms is also an alternative RSM.

$$Y = b_0^{'} + b_1^{'}X_0^{'} + b_2^{'}X_2^{'} + b_3^{'}X_3^{'} + b_{12}^{'}X_1^{'}X_2^{'} + b_{13}^{'}X_1^{'}X_3^{'} + b_{23}^{'}X_2^{'}X_3^{'} + b_{123}^{'}X_1^{'}X_2^{'}X_3^{'}, \qquad (9.16)$$

in which $b_{12}^{'}$, $b_{13}^{'}$, $b_{23}^{'}$, and $b_{123}^{'}$ are additional undetermined coefficients.

Step 4: Perform experiment to obtain the response

The FEA is performed at the 8 design points given in Table 9.1 and Fig. 9.3. The calculated stresses $Y^{\exp}$ are listed in Table 9.2.

Table 9.2 DOE Matrix

Experiment	$X_1^{'}$	$X_2^{'}$	$X_3^{'}$	$Y^{\exp}$
1	-1	-1	-1	60
2	1	-1	-1	72
3	-1	1	-1	54
4	1	1	-1	68
5	-1	-1	1	52
6	1	-1	1	83
7	-1	1	1	45
8	1	1	1	80

The experimental results are also plotted in Fig. 9.3, where the quantities within the circles are the response variable Y calculated from the FEA simulation.

Step 5: Determine the unknown coefficients of the RSM

The undetermined coefficients are solved out with the least square difference between the predicted response from Eqs. 9.15 or 9.16 and the experimental results $Y^{\exp}$. The model for solving the undetermined coefficients is given by

$$Min \sum_{i=1}^{8}\left[Y_i^{\exp} - \left(b_0^{'} + b_1^{'}X_0^{'} + b_2^{'}X_2^{'} + b_3^{'}X_3^{'}\right)\right]^2 \qquad (9.17)$$

for the leaner model in Eq. 9.15, and

$$Min \sum_{i=1}^{8}\left[Y_i^{\exp} - \left(b_0^{'} + b_1^{'}X_0^{'} + b_2^{'}X_2^{'} + b_3^{'}X_3^{'} + b_{12}^{'}X_1^{'}X_2^{'} + b_{13}^{'}X_1^{'}X_3^{'} + b_{23}^{'}X_2^{'}X_3^{'} + b_{123}^{'}X_1^{'}X_2^{'}X_3^{'}\right)\right]^2, \qquad (9.18)$$

for the quadratic model in Eq. 9.16.

Eqs. 9.17 and 9.18 can be solved by optimization. Alternatively, based on the above models, the coefficients can also be calculated as follows.

The first coefficient is the average of the responses from the experiment,

$$b_0 = \frac{1}{8}\sum_{i=1}^{8} Y_i^{exp} = 64.25.$$

The coefficients of the first order terms $b_1^{'}$, $b_2^{'}$, and $b_3^{'}$ are calculated from the main effects. The main effect E_i of variable $X_i^{'}$ is the average change when $X_i^{'}$ change from -1 to +1 while other variables remain unchanged at their averages. Therefore the main effect E_i is computed as the difference between the average value P_i^+ of the response at the $X_i^{'}$ high level (+1) and the average value P_i^- of the response at the $X_i^{'}$ low level (-1). For $X_1^{'}$, $b_1^{'}$ is computed as follows.

The average response at $X_1^{'}$ high level (+1) (see Fig. 9.4) is,

$$P_{1+} = \frac{1}{4}(Y_2^{\exp} + Y_4^{\exp} + Y_6^{\exp} + Y_8^{\exp}) = \frac{1}{4}(72 + 68 + 83 + 80) = 75.75,$$

and the average response at $X_1^{'}$ low level (-1) is,

$$P_{1-} = \frac{1}{4}(Y_1^{\exp} + Y_3^{\exp} + Y_5^{\exp} + Y_7^{\exp}) = \frac{1}{4}(60+54+52+45) = 52.75.$$

The main effect of $X_1^{'}$ is given by

$$E_1 = P_{1+} - P_{1-} = (75.75 - 52.75) = 23.$$

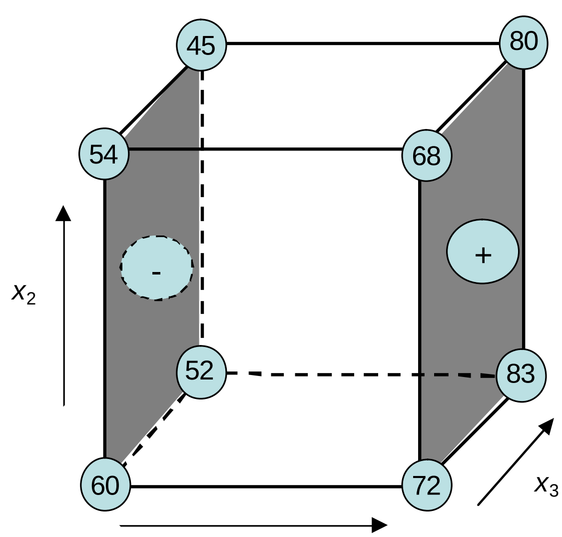

Figure 9.4 The Main Effect of $X_1^{'}$

The coefficient of $X_i^{'}$ is half of the main effect E_i , namely,

$$b_i = \frac{1}{2} E_i \,. \tag{9.19}$$

Therefore,

$$b_1 = E_1 / 2 = 11.5 .$$

Similarly, $b_2^{'}$ and $b_3^{'}$ are calculated as follows.

$$P_{2+} = \frac{1}{4}(Y_3^{\exp} + Y_4^{\exp} + Y_7^{\exp} + Y_8^{\exp}) = \frac{1}{4}(54+68+45+80) = 61.75 \text{ (see Fig. 9.5)},$$

$$P_{2-} = \frac{1}{4}(Y_1^{\exp} + Y_2^{\exp} + Y_5^{\exp} + Y_6^{\exp}) = \frac{1}{4}(60+72+52+83) = 66.75,$$

$$E_2 = P_{2+} - P_{2-} = 61.75 - 66.75 = -5 .$$

Hence,

$$b_2 = E_2 / 2 = -2.5 .$$

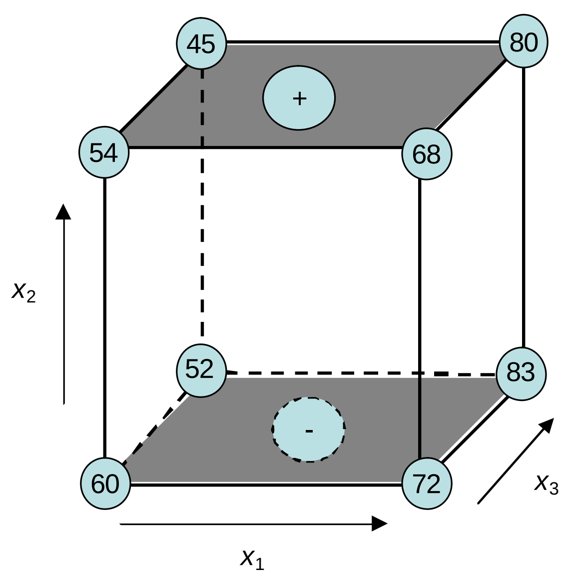

Figure 9.5 The Main Effect of $X_2^{'}$

$$P_{3+} = \frac{1}{4}(Y_4^{\exp} + Y_5^{\exp} + Y_6^{\exp} + Y_7^{\exp}) = \frac{1}{4}(52+83+45+80) = 65 \text{ (see Fig. 9.5)},$$

$$P_{3-} = \frac{1}{4}(Y_1^{\exp} + Y_2^{\exp} + Y_3^{\exp} + Y_4^{\exp}) = \frac{1}{4}(60+72+54+68) = 63.5,$$

$$E_3 = P_{3+} - P_{3-} = 63.5 - 65 = -2.5.$$

Hence,

$$b_3 = E_3 / 2 = -1.25.$$

(9.32)

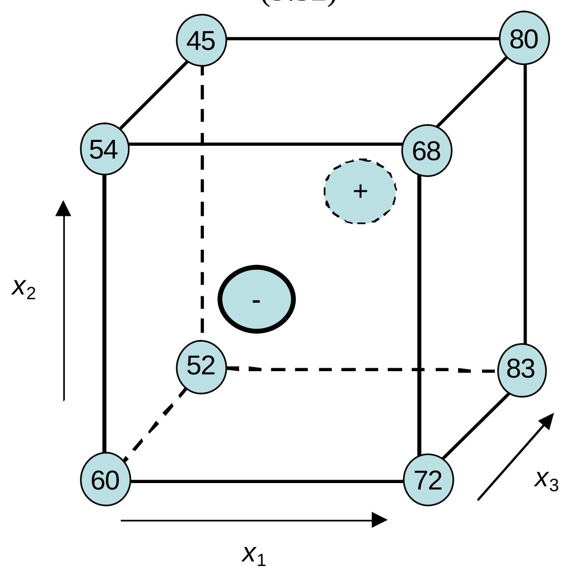

Figure 9.6 The Main Effect of $X_3^{'}$

Therefore, the linear RSM is obtained, which is given by

$$Y = 64.25 + 11.5X_1^{'} - 2.5X_2^{'} + 0.75X_3^{'}. \quad (9.20)$$

Using Eq. 9.14, $X_1^{'}$, $X_2^{'}$, and $X_3^{'}$ are transformed into original variables, and then the RSM is rewritten in terms of the original variables as below.

$$Y = 64.25 + 11.5\left[\frac{2(X_1 + 160)}{170 + 160} - 1\right] - 2.5\left[\frac{2(X_2 + 20)}{40 + 20} - 1\right] + 0.75X_3.$$

Or

$$Y = 64.73 + 0.0697X_1 - 0.0833X_2 + 0.75X_3. \quad (9.21)$$

In the quadratic model in Eq. 9.16, the interaction terms are included. If treating $X_1^{'}X_2^{'}$, $X_1^{'}X_3^{'}$, $X_1^{'}X_3^{'}$, and $X_1^{'}X_2^{'}X_3^{'}$ as three new individual variables, we can use the same approach as we did above to calculate the coefficients of the interaction terms. For this purpose, Table 9.3 is then rewritten in order to include the new variables (the interactions). The table is given below.

Table 9.4 Experimental Results

Experiment	$X_1^{'}$	$X_2^{'}$	$X_3^{'}$	$X_1^{'}X_2^{'}$	$X_1^{'}X_3^{'}$	$X_2^{'}X_3^{'}$	$X_1^{'}X_2^{'}X_3^{'}$	Y^{exp}
1	-1	-1	-1	1	1	1	-1	60
2	1	-1	-1	-1	-1	1	1	72
3	-1	1	-1	-1	1	-1	1	54
4	1	1	-1	1	-1	-1	-1	68
5	-1	-1	1	1	-1	-1	1	52
6	1	-1	1	-1	1	-1	-1	83
7	-1	1	1	-1	-1	1	-1	45
8	1	1	1	1	1	1	1	80

The calculations are given below.

$$P_{12+} = \frac{1}{4}(Y_1 + Y_4 + Y_5 + Y_8) = \frac{1}{4}(60 + 68 + 52 + 80) = 65.$$

$$P_{12-} = \frac{1}{4}(Y_2 + Y_3 + Y_6 + Y_7) = \frac{1}{4}(72+54+83+45) = 63.5.$$

$$E_{12} = P_{12+} - P_{12-} = (65 - 63.5) = 1.5.$$

$$b_{12} = E_{12}/2 = 0.75.$$

$$P_{13+} = \frac{1}{4}(Y_1 + Y_3 + Y_6 + Y_8) = \frac{1}{4}(60+54+83+80) = 69.25.$$

$$P_{13-} = \frac{1}{4}(Y_2 + Y_4 + Y_5 + Y_7) = \frac{1}{4}(72+68+52+45) = 59.25.$$

$$E_{13} = P_{13+} - P_{13-} = (69.25 - 59.25) = 10.$$

$$b_{13} = E_{13}/2 = 5.$$

$$P_{23+} = \frac{1}{4}(Y_1 + Y_2 + Y_7 + Y_8) = \frac{1}{4}(60+72+45+80) = 64.25.$$

$$P_{23-} = \frac{1}{4}(Y_3 + Y_4 + Y_5 + Y_6) = \frac{1}{4}(54+68+52+83) = 64.25.$$

$$E_{23} = P_{23+} - P_{23-} = (64.25 - 64.25) = 0.$$

$$b_{23} = E_{23}/2 = 0.$$

$$P_{123+} = \frac{1}{4}(Y_2 + Y_3 + Y_5 + Y_8) = \frac{1}{4}(72+54+52+80) = 64.5.$$

$$P_{123-} = \frac{1}{4}(Y_1 + Y_4 + Y_6 + Y_7) = \frac{1}{4}(60+68+83+45) = 64.$$

$$E_{123} = P_{123+} - P_{123-} = (64.5 - 64) = 0.5.$$

$$b_{123} = E_{123}/2 = 0.25.$$

Therefore, the nonlinear RSM is given by

$$Y = 64.25 + 11.5X_1^{'} - 2.5X_2^{'} + 0.75X_3^{'} + 0.75X_1^{'}X_2^{'} + 5X_1^{'}X_3^{'} + 0.25X_1^{'}X_2^{'}X_3^{'}. \tag{9.22}$$

Using Eq. 9.14, we can obtain the RSM in terms of the original variables below.

$$\begin{aligned} Y &= 64.25 + 11.5\left[\frac{2(X_1+160)}{170+160} - 1\right] - 2.5\left[\frac{2(X_2+20)}{40+20} - 1\right] + 0.75X_3 \\ &+0.75\left[\frac{2(X_1+160)}{170+160} - 1\right]\left[\frac{2(X_2+20)}{40+20} - 1\right] + 5\left[\frac{2(X_1+160)}{170+160} - 1\right]X_3 \\ &+0.25\left[\frac{2(X_1+160)}{170+160} - 1\right]\left[\frac{2(X_2+20)}{40+20} - 1\right]X_3 \end{aligned}$$

Or,

$$Y = 64.74 - 0.0682X_1 - 0.0841X_2 - 0.5985X_3 + 0.0002X_1X_2 + 0.0303X_2X_3. \tag{9.23}$$

Step 6: Use the response surface model for uncertainty analysis.

After the RSM is obtained, the original expensive FEA model will be replaced for uncertainty analysis and probabilistic design. Step 6 will be demonstrated in the following example.

Example 9.5 – RSM based reliability analysis of the cantilever beam

We will use RSM to solve the reliability analysis for the cantilever beam given in Examples 7.2 and 9.1 In this problem, in addition to the random variables $P_x \sim N(500,40)lb$ and $P_y \sim N(1000,80)lb$, the Young's modulus E is also considered normally distributed with $E \sim N(30\times10^6, 10\times10^5)psi$. Next, we will create a RSM in terms of the three random variables by using the 2-level full factorial design.

Step 1: Determine the design variables and response variables

The design variables are the three random variables P_x , P_y , and E . The response variable is the tip deflection of the beam, *Y*, which is given by

$$Y = \frac{4L^3}{Ewt}\sqrt{\left(\frac{P_y}{t^2}\right)^2 + \left(\frac{P_x}{w^2}\right)^2} ,$$

in which all the constants are the same as in Example 7.2.

Step 2: Determine the design variable bounds

The lower and upper bounds are determined by " $3s$ " principle. The bounds are given by

$$[P_{x\min}, P_{x\max}] = \left[m_{P_x} - 3s_{P_x},\ m_{P_x} + 3s_{P_x}\right] = [380,\ 620],$$

$$\left[P_{y\min}, P_{y\max}\right] = \left[m_{P_y} - 3s_{P_y},\ m_{P_y} + 3s_{P_y}\right] = [760,\ 1240],$$

and

$$[E_{\min}, E_{\max}] = [m_E - 3s_E,\ m_E + 3s_E] = \left[27\times10^6, 33\times10^6\right].$$

Step 3: Plan the experiment

Two levels for each design variable are considered for the experiments. The intended RSM is given by

$$Y = b_0^{'} + b_1^{'}P_x^{'} + b_2^{'}P_y^{'} + b_3^{'}E^{'} + b_{12}^{'}P_x^{'}P_y^{'} + b_{13}^{'}P_x^{'}E^{'} + b_{23}^{'}P_y^{'}E^{'} + b_{123}^{'}P_x^{'}P_y^{'}E^{'} .$$

The DOE matrix is given in Table 9.5, and the values of the random variables are shown in brackets.

Step 4: Perform experiment to obtain the response

The tip deflections at the 8 design points are calculated and are listed in Table 9.5.

Table 9.5 DOE Matrix and Experimental Results

Experiment	$P_x^{'}$ (lb)	$P_y^{'}$ (lb)	$E^{'}$ (psi)	$Y^{\exp}$ (in)
1	-1 (380)	-1 (760)	-1 (27×10^6)	1.9669
2	1 (620)	-1 (760)	-1 (27×10^6)	3.0021
3	-1 (380)	1 (1240)	-1 (27×10^6)	2.2704
4	1 (620)	1 (1240)	-1 (27×10^6)	3.2092

5	-1 (380)	-1 (760)	1 (33×10^6)	1.6093
6	1 (620)	-1 (760)	1 (33×10^6)	2.4563
7	-1 (380)	1 (1240)	1 (33×10^6)	1.8576
8	1 (620)	1 (1240)	1 (33×10^6)	2.6257

Step 5: Determine the unknown coefficients of the response surface model

The unknown coefficients are computed using the equations discussed above, and the RSM is then given by

$$\begin{aligned} Y &= 2.3747 + 0.4486P_x^{'} + 0.1160P_y^{'} - 0.2375E^{'} \\ &-0.0219P_x^{'}P_y^{'} - 0.0116P_x^{'}E^{'} - 0.0449P_y^{'}E^{'} + 0.0022P_x^{'}P_y^{'}E^{'} . \end{aligned}$$

This model is for the normalized random variables at the scale of [-1, 1]. Eq. 9.14 can be used to convert the model into the original random variables.

Step 6: Use the response surface model for uncertainty analysis

Since the RSM is cheap to compute now, Monte Carlo simulation with a large sample size of 10^6 is used to compute the probability of failure $p_f = P\{g = D_0 - Y < 0\}$, where D_0 is the allowable deflection and $D_0 = 3in$. The calculated probability of failure based on the RSM is $p_f = 0.000175$. To confirm the result, the same size of Monte Carlo simulation is also performed with the original performance function and the result is $p_f = 0.000157$. It is noted that with only 8 function evaluations (for constructing the RSM), the RSM based reliability analysis method produces a very accurate reliability result.

Herein, only a simple two-level full factorial design is discussed. To make a RSM more accurately represent the original function, higher levels can be used. For example, in a 3-level design full factorial design, for each design variable, in addition to the two end points, the center point is also used in constructing the RSM. With higher level DOE, a high order RSM, for example, a cubic polynomial, can be used to achieve accurate results. However, more levels and a higher order RSM need more experiments. To make reasonable good trade-off between accuracy and efficiency, a fractional factorial design may be considered, where not all the combinations of the design variables levels will be considered. The complete methodology of RSM can be found in many statistical experiment design books.

To better capture the high nonlinearity and achieve higher accuracy, other DOE methods such as Kriging method, MARS (Adaptive Regression Splines), and radial basis functions have been developed and have been increasingly used in engineering applications. A vast amount of literature is available in this subject.

9.5 Conclusion

The task of uncertainty analysis is to identify the probabilistic characteristics of performance functions. The probabilistic characteristics of performance functions will be used in the design stage for managing and mitigating the effects of input uncertainty on the performance. The probabilistic characteristics of performance include the moments (mean, standard deviation, etc.), percentile values, the probability of failure, reliability, and probability distributions. Depending on different applications, different probabilistic characteristics will be needed. For example, robust design needs mean and standard deviation, reliability-based design needs reliability, and risk analysis needs the probability of failure.

In Chapters 7, 8 and 9 (this chapter), we discussed the commonly used uncertainty analysis methods. It should be noted that there is no universal uncertainty analysis method that suits all the situations in engineering analysis and design. For different problems, we may choose different uncertainty analysis methods. When we consider choosing uncertainty analysis methods, accuracy and efficiency are the major concerns. There always exists a conflict between accuracy and efficiency. The robustness, which measures if a method can successfully identify the uncertainty analysis solutions, is also a factor of consideration. A comparison among the methods we have discussed is given in Table 9.6 [3].

Table 9.6 Comparison of Uncertainty Analysis Methods

	MCS	**FORM**	**SORM***	**FOSM***	**Worst Case Analysis**	**RSM**
Requires input distributions	Yes	Yes	Yes	Distributions or the first two moments	Intervals	May or may not
Deals with correlation	Yes	Yes	Yes	Yes	No	Yes
Requires derivative of performance function	No	Yes	Yes	Yes	Yes	May or may not
Efficiency	Needs a large number of function evaluations, especially when the probability is high.	Efficient for small or moderate number of random variables; efficient than SOME; efficiency decreases with a large number of random variables.	Efficient for small number of random variables; needs 2nd derivatives; efficiency decreases with a large number of random variables.	Very efficient.	Very efficient.	Efficient with small or moderate number of design variables. Huge computational demand when the number of random variables is large.
Capability and accuracy	Gives accurate solutions when enough samples are used; can generate the complete distribution.	The accuracy depends on the performance function and input distributions; generally, more accurate than moment matching and RSM.	The accuracy depends on the performance function and input distributions; generally, more accurate than FORM.	Simple to use, but generally not accurate.	Approximation on the bunds of a performance function.	The accuracy depends on how accurately the RSM represents the performance function; may result in errors.
Robustness	Very robust (can always find the solution.)	The MPP search may not converge.	The MPP search may not converge.	Robust	Robust	Robust if MCS is used; the MPP search may not converge if FORM or SORM is used.

* If a gradient-free optimization algorithm is used, no derivative is needed.

Reference

[1] Ferson S., Joslyn, C.A., Helton, J.C., Oberkampf, W.L., and Sentz, K., 2004, "Summary from the Epistemic Uncertainty Workshop: Consensus amid Diversity," Reliability Engineering and System Safety, 85 (1-3), pp. 355–369.

[2] Du, X., Sudjianto, A., and Huang, B., "Reliability-Based Design under the Mixture of Random and Interval Variables," in press, ASME Journal of Mechanical Design, 2006.

[3] Du, X. and Chen, W., "Towards a Better Understanding of Modeling Feasibility Robustness in Engineering," ASME Journal of Mechanical Design, Vol. 122, No. 4, pp. 357-583, 2000.