A joint of a mechanism can be considered as a journal bearing. As shown in Fig. 1, the radius of the bearing r_B is greater than that of the journal r_J . Their difference is called a clearance, and $r = r_B - r_J$. As indicated in Fig. 2, the position (X, Y) of the center of the journal relative to the center of the bearing is within a circle of radius r. This circle is called a clearance circle.

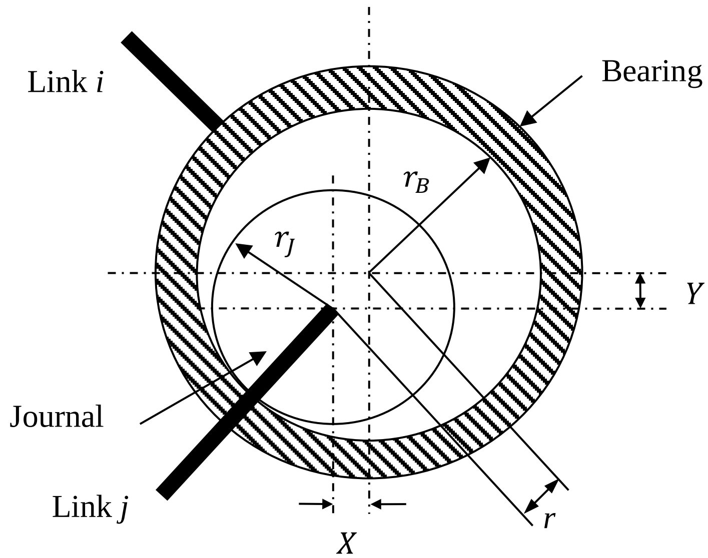

Fig. 1 Joint clearance

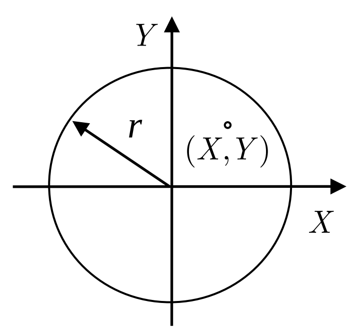

Fig. 2 Clearance circle

Assume that the clearance is known and that the position (X, Y) is uniformly distributed within the clearance circle $\sqrt{X^2 + Y^2} \leq r$. Determine

(1) The joint PDF and CDF of *X* and *Y*

(2) The PDFs of *X* and *Y*

(3) The means of *X* and *Y*

(4) The variances of *X* and *Y*

(5) The coefficient of correlation between of *X* and *Y*

Solution

(1) The joint PDF and CDF of X and Y

The position (X, Y) is uniformly distributed within the clearance circle $\sqrt{X^2+Y^2} \leq r$, and the PDF is constant within the clearance circle $\sqrt{X^2+Y^2} \leq r$.

$$f_{X,Y}(x,y) = \begin{cases} K & x^2+y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

where K is a constant.

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dxdy = 1$$

$$K \iint_{x^2+y^2 \leq r^2} dxdy = 1$$

$$K\pi r^2 = 1$$

$$K = \frac{1}{\pi r^2}$$

Therefore

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2+y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

(2) The PDFs of X and Y

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2} \, dy = \frac{2\sqrt{r^2-x^2}}{\pi r^2} \text{ }, -r \leq x \leq r$$

Therefore

$$f_X(x) = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & , -r \leq x \leq r \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & , -r \leq y \leq r \\ 0 & \text{otherwise} \end{cases}$$

(3) the mean of X is

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x \, f_X(x) dx = \int_{-r}^{+r} x \frac{2\sqrt{r^2-x^2}}{\pi r^2} dx = 0$$

Similarly,

$$\mu_Y = 0$$

(4) The variances of *X* and *Y*

$$\sigma_X^2 = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx = \int_{-r}^{+r} x^2 \frac{2\sqrt{r^2 - x^2}}{\pi r^2} dx = \frac{r^2}{4}$$

Similarly,

$$\sigma_Y^2 = \frac{r^2}{4}$$

(5) The coefficient of correlation between of *X* and *Y*

$$Cov(X,Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_X)(y - \mu_Y) f_{X,Y}(x,y) dxdy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - 0)(y - 0) \frac{1}{\pi r^2} dxdy$$

$$= \frac{1}{\pi r^2} \int_{-r}^{+r} x \left[\int_{-\sqrt{r^2 - x^2}}^{\sqrt{r^2 - x^2}} y \, dy \right] dx$$

$$= 0$$